PAN SCIENCE SYSTEMS
12121 Wilshire Blvd., Suite 215
Los Angeles, California 90025

TECHNICAL REPORT TR 2010

THE IMPROVEMENT OF EFFICIENCY IN THE NUMERICAL COMPUTATION OF ORBIT TRAJECTORIES

(NASA-CR-132825) THE IMPROVEMENT OF EFFICIENCY IN THE NUMERICAL COMPUTATION OF ORBIT TRAJECTORIES (Pan Science Systems, Los Angeles, Calif.) 232 p HC $13.75 CSCL 22A G3/30

N74-10774

Unclas
20855

By:

J. Dyer

R. Danchick

S. Pierce

R. Haney

This document was produced by PSS in performance of Contract NAS 5-11940 for Goddard Space Flight Center

Final Report

TABLE OF CONTENTS

Table of Contents

LIST OF FIGURES

List of Figures

0. INTRODUCTION AND CONCLUSIONS

INTRODUCTION

This Report describes all analysis, system design, programming and evaluation of results of work done in performance of Contract NAS5 11940 for the System Development and Analysis Branch, Mission Support Computing and Analysis Division, Goddard Space Flight Center, (NASA) Greenbelt, Maryland.

Contract work continues the effort of developping, testing and implementing generalized numerical integration methods. Background material and basic results are contained in Reports [2] and [6]. Underlying and motivating all work is the concept of efficiency in orbit computation. Efficiency is measured by the amount of computing machine time required to generate a solution trajectory with given approximate accuracy. Efficiency depends on the machine, the code, the accuracy requirement, the integrator, and the equations of motion and the problem among other variables. The effort is directed towards improvement of efficiency. This involves both the reduction of local truncation error and error propagation.

The objectives of the work embodied in this contract were:

- - - To complete the final evaluation of generalized methods: cyclic composite methods, matrix type methods and off-grid methods, including the use of repeated off-grid points and the use of third derivative values in the difference equations, in their application to orbit computation.

- - - To begin the study of the interaction of different formulations of the equations of satellite motion with different numerical integration algorithms from the point of view of efficiency in generating the solution of basic orbit types. The final goal in this kind of study is a guide to the selection of formulation/integrator combinations for the orbit types.

A study of the effect on local and propagated error of transforming equations of motion and integrators loads to an examination of the meaning of numerical stability for the Keplerian orbit problem as well as asymptotic error growth rates.

- - - To continue the effort of developping efficient integrators based on the extension to second-order equations of well known results pertaining to the reduction of propagated error in the numerical solution of first-order differential equations. Experiments in optimization were combined with an guided analytical results.

- - - To consider the problem of high-order iteration.

Resume of Results

The error in multistep solution of a system of ordinary differential equation has been modelled more exactly than has heretofore been accomplished. This has been done for the purpose of analyzing the interaction of integrator and formulation of the equations of motion. The matrix form of the multistep difference operator given by Gear [8] has been found to be a natural tool for this purpose. Prior analysis had been restricted to linear or linearized systems. Present analysis begins at that point (local linearization) and solves the system of error equations

$$e_{n+1} = S_n e_n + b_n$$

at each step time where S_n, the local error operator, is a matrix depending on predictor and corrector coefficients, the Jacobian matrix of the system and the step-size h. The vector b_n represents local truncation error.

This type of error analysis results in the explicit determination of a composite error operator propagating an initial error, e_o, and a term describing local error and its propagation. Thus these two facets of accumulated error can be studied separately.

Derivation of a composite error operator is particularly valuable in the special case of Keplerian motion. Step-sizes may be selected which divide the period and global numerical stability, eventual growth or damping of an initial parasitic error solution, can then apparently be studied in terms of the non-principal eigenvalues of the composite error operator, i.e., the operator which moves the error through exactly one revolution. Moreover, the general stability problem can be approached by an analysis of the canonical form as well as all eigenvalues of the composite operator. Inferences can no doubt be made concerning the stability of multistep numerical integration configuration applied to perturbed trajectories in terms of an average period.

In the Keplerian case, the local error operator (Cowell/Class II) has eigenvalues equal to the (principal and non-principal) roots of the characteristic polynomial resulting from the usual linear analysis. The composite operator has apparently four eigenvalues of unity corresponding to the double negative roots of the Jacobian of the system and two real positive mutually reciprocal roots, near unity, corresponding to the single positive eigenvalue of the system Jacobian.

In the simple linear analysis, it is well known that principal roots approximate certain exponential functions of h, and that the deviation from the exponentials represents truncation

error. In the present composite analysis, it is not clear what function the eigenvalues $\neq 1$ are approximating. These eigenvalues presumably reflect both the instability of the equations and local truncation error. These roots (of the composite error operator) are given in the text for algorithms PE and PECE.

The capabilities of the multistep Analyzer are demonstrated in the text in the analysis of local and propagated error with various algorithms and trajectories including the classical equations solved as both first order and second order equations on a portion of a high drag trajectory. Variation of parameter integration is capable of analysis if the system Jacobian is input although the algorithm using a terminal partial evaluation is not yet implemented.

It is shown in the report that change of basis [9] applied to a matrix type multistep operator can produce an equivalent operator which yields considerably reduced error in a simple computation applied to a first order equation. In connection with Keplerian orbital motion two different basis changes were studied. The basis corresponding to the Adams, Class II, 12 value method is of type 2-10, i.e., the basis is comprised, of two back function values and ten back values of the second derivative. The two other bases investigated were of types 4-8 and 6-6. Transforms of the Adams method were derived relative to

bases and computation done. Neither of the transformed methods was as efficient as the original Adams method as tested on a two-hour circular trajectory.

Transformed methods involve corrections to back points. Experiments were performed in optimizing correction coefficients. Efficiency can be increased by this means but was not increased to the level of the Adams method.

R. Danchick has derived a class of methods using a Nordsieck type basis. A 13th order method of this kind (Class II) was tested against the Adams method of the same order. Propagated error in Danchick's method was slightly, but not significantly, larger. Step-size change in these methods is, of course, very easily accomplished. If varistep algorithms will be continued in use at Goddard Space Flight Center it is recommended that these methods be seriously considered for use. Derivation of Danchick's methods for first-order equations appears in Appendix B.

Basic results concerning efficiency and minimization of propagated error for first-order equations have been extended to second-order equations in Appendix A. This effort extends the scope of the contract work. By application of this theory, many new methods have been constructed. These methods are defined

in Section 2 and results are given here of some computations of one week of a circular trajectory with period of two hours. The approximate length of the position error vector is given in meters. The Störmer predictor of corresponding order is used throughout. Orders 7 through 13 were considered.

<u>h = 100 secs., Alg. PEC</u>

method	order	error
RW(.3)	11	.0004
W(.6)	"	.0016
W(.7)	"	.0030
W(.8)	"	.0030
W(.8)	10	.0120
W(.7)	10	.0140
RW(.4)	10	.0150
W(.6)	10	.0160
Adams	11	.0200
Adams	7	476.8000
Adams	12	unstable

<u>h = 300 secs., Alg. PECE</u>

method	order	error
WA(.3)	13	17.24
W(.2)	13	20.70
RW(.1)	13	23.66
WB(.1)	13	24.50
Adams	13	39.80
W(.7)	11	64.60
WA(.3)	12	140.00
W(.7)	10	148.70
Adams	12	604.96
Adams	11	1399.28
Adams	9	36581.38

Higher order Adams methods than are shown here are stable at h = 300 (Alg. PECE). The advantage of optimization is diminished at large step-size, using two corrections. In this case,

optimization, which involves moving non-principal roots at h = 0 away from the complex plane, normally reduces stability. If the order of the Adams method is taken large enough so that little unused stability is present, optimization is difficult.

Perhaps the most important algorithm is PECE* where, terminally, the central force only is reevaluated. Some experimentation was performed with this algorithm by the simple expedient of subtracting out the predicted central force and adding in the corrected one. Computations were done at the same trajectory, with perturbations, for about ten revolutions. Methods used were the 9th order Adams and W(.7) methods. These reults were rather confusing. Method W(.7) proved to be somewhat more efficient at practical step-sizes (h = 100 secs., 120 secs.). At h = 130 secs., the Adams method yielded over 82 km. of error while method W(.7) was unstable. Method W(.7) also showed instability at very small step-sizes (e.g. h = 50 secs.). Possibly the primitive manner in which the E* function was constructed has invalidated results, although the reason for this is unclear to the principal investigator. It is recommended that all new methods be tested at GSFC using the partial evaluation algorithm. All coefficients will be delivered.

Experiments were conducted with a method whose non-principal roots at h = 0 were those of W(.7) order 13. The order of the method, however, was diminished by 2 and the (b_i) coefficients determined both by order conditions (order 11) and by formal minimization of various simple models of truncation error. These methods yielded slightly smaller errors, in all tests, than the original method: W(.7), order 13. This work extended the contract scope.

It is felt that, further research in integrators should be directed along these lines. More sophisticated truncation error models should be used depending on some or all of the following items: The orbit type, the orientation of the orbit plane, approximate position, and the coordinate being integrated. Computation of a given orbit would entail use of a small set of integrators used cyclically. Any further significant increase in integrator efficiency will probably require use of actual trajectory information. The most significant advance in orbit computation algorithms to date has been the factorization of the integrator into Class I and Class II components. In this, the actual nature of the motion has guided the model. The ideas presented above are a natural extension of this notion.

A coefficient generator was produced which provided the capability of using a repeated off-grid acceleration $\ddot{y}_{n-i-e}$ in the difference equation, of using third derivative values at

step times (often easily available) in the difference equation, or any combination of these items. The inclusion of the repeated off-grid point capability was an extension of the original scope of the work.

Details of the coefficient generator and test results can be found in Section 7. The program automatically performs tests by integration of a simple trajectory of the method specified by input and derived is stable at h = 0. The use of exact, analytical predictors yielded excellent test results on a short circular arc. These results were given in an interim report. However, tests, using a fixed basis, 11th order, quasi-Hermite predictor yielded no method which surpassed in efficiency the Adams method. The reason may lie in the prediction problem, but it is judged that the reason is contained in the discussion of general purpose (except for the Class I/Class II structure) integrators versus highly special purpose integrators.

A program was designed and written for the purpose of converting any traditional (on-grid) Class II difference equation in the usual ordinate form to the difference form and from the difference form to the summed difference form and to the summed ordinate form. The program converts both predictors and correctors. Summed methods were produced and tested in a two-body integrator. The position error propagated by summed difference

equations was, as expected, less than the error in the unsummed method. However, the difference was not as great as expected, due probably to the fact that in the implementation, sums were initialized by use of the corrector formulas rather than computing them at a central point with a central difference formula and back-dating them, with the recursion formulas, to the applicable time. Coefficients were validated by converting Adams methods and comparing coefficients with those given in the literature.

Dr. Pierce has continued his analysis of the cyclic composite methods, partially supported under this contract and under independent grant.

Four basic types of iterative starting methods are discussed in the report in terms of convergence and rate of convergence. It is shown essentially that the maximal order obtainable in a difference equation by use, for example, of one off-grid derivative value cannot be maintained in a system of independent difference equations necessary for iterative starting. Other off-grid points must be introduced. Therefore there does not appear to be any advantage, as far as order is concerned, in departing from the on-grid framework using a larger step-number and/or smaller step-size to increase order.

Comparison of the efficiency of the various iterations discussed is an interesting problem, but not considered here.

Interpolation for position, necessary in the orbit determination process can be done with high-order quasi-Hermite interpolation formulas. The coefficients in two such formulas have been accurately computed and are given in this report.

Coefficients of summed ordinate forms are not accurate to the full possible sixteen digit precision. This is due to the fact that the input is sixteen digit output from Program A/B. More precision can readily be obtained by requiring Program A/B, which uses the CDC 6000 series machine, to punch more precise card output. Work on the summed forms was beyond the original scope of the contract.

S. Pierce has had primary responsibility for sections 3 and 5. R. Haney performed some analysis in connection with Section 1 and was responsible for a major portion of the programming for the contract effort.

1. Other Formulations - Integrator Interactions

The purpose of this task is to study the interactions of various formulations of the equations of satellite motion with various numerical integration algorithms. In order to do this, the local truncation error is studied independently (as independently as possible) from the error propagation characteristics of the formulation/integrator pair. The final objective of the study is to provide an analytical tool for selecting efficient formulation/integrator pairs for orbit types which includes the optimization of the numerical integrator. It is important to understand what effect the transformation from the classical equation to other formulations has on error propagation, to study the effect of solving the classical equations as first-order equations on high drag trajectories, for example, and to investigate the efficiencies of the various integration algorithms.

Partial motivation for the study consists in the many, and sometimes confusing, opinions and viewpoints voiced by advocates of one method or another. See, for example, [1, 3, 5]. In addition, S. S. Dallas has recently published interesting data on the nature of the growth (and reduction) of accumulated error in a Mars orbiter trajectory in the first few revolutions. This kind of information can quite possibly be of use in the design of better orbit computation procedures.

One reason for the lack of agreement in these matters is the complexity of the problem and the varied technical background of those re-

porting results. Efficiency depends heavily on accuracy requirements, on the integrator and the integrator algorithm structure. These factors are often not given sufficient attention.

1.1 Error analysis in multistep methods

Much work has been done in the past towards the analysis of local errors and error propagation in the solution of ordinary differential equations by multistep methods. It is usual to apply the analysis of propagated error to the greatly simplified problem in which truncation error is assumed constant at each step and the Jacobian matrix, $\partial F/\partial y$, of the given system

$$\dot{y} = F(y,t)$$

is constant.

The objective of the current analysis is to model the propagated error more exactly in the multistep solution -- i.e., the analysis and the associated program (based on C. W. Gear's matrix formulation of the multistep operator [8]) permits the variable error propagation matrix, as well as the local truncation error, to be computed at each step. In this way, the true first-order effects of the error propagation are obtained. The propagated error vector, in general, contains components corresponding to predictor error and predictor-corrector error for each variable and at each time point involved in the difference equations.

The analysis and the implementation are described more fully in the following section in the general multistep methods analyzer. Principal application is, of course, to the orbit problem. In this application, the causes of error growth can be isolated and studied in the perturbed problem and, in addition, global, non-linear stability criteria are available relative to the Keplerian problem. The composite error propagation operator, which propagates an arbitrary initial error vector, through n steps, is computed in the normal operation of the program. If $nh = p$, (i.e., the step size divides the period), then the eigenvalues of the composite operator which propagates error through a revolution can be computed on option and, as usual, classified as principal or non-principal roots. In this way, the numerical stability analysis can be extended from the linear to the non-linear Keplerian case. The effect of instabilities of the equations can be studied as well as asymptotic error growth rates. Moreover, inferences about numerical stability, instability, in perturbed trajectory applications can perhaps be made.

The program is now capable of error analysis of the following cases:

Cowell: Class II; PE, PEC, PECE*

Cowell: Class I; PECE*

Cowell: Class I/Class II; PECE*

VOP: PE, PEC

Here, and elsewhere, E* indicates a partial evaluation.

The scope of this task has been increased beyond the original intention -- which was simply to study the effect of change of formulation and integrator on the local error and on local propagation properties. It was later judged important to provide a complete solution to the error equations, with a norm and complete spectrum of the local and composite operators involved, norms of local and propagated error vectors and a plot routine, as well as an analysis of asymptotic error growth. The equations have been derived for a local error operator for VOP/PECE*, but the analysis has not been implemented. It was felt more important to implement Class I and Class I/Class II for Cowell, PECE*.

1.2 General multistep methods analyzer (The program)

The purpose of the General Multistep Method Analyzer is to analyze the errors of general multistep numerical integration methods, i.e., to compare methods, step-sizes, algorithms, and formulations for efficiency studies, and to relate errors to the controllable variables.

In the past, one of the major limitations of research in general multistep methods has been in the use of the linear, scalar, constant-coefficient differential equation

$$y^{(j)} = \lambda y, \qquad j = 1 \text{ or } 2 \tag{1-1}$$

for a model of the behavior of errors of numerical integration. In actual practice, for orbit computations, there are generally three or more

components of errors, and the actual differential equations integrated are similar to

$$y^{(j)} = \lambda y, \qquad j = 1 \text{ or } 2 \tag{1-2}$$

where y is a vector and λ is a matrix of variable entries, and t is sometimes a function of a new independent variable.

By applying the difference operators of the numerical methods to the actual differential equations integrated, we obtain a vector difference equation of the form

$$e_n = S_n e_{n-1} + b_n, \qquad (n = 1,2,3,\ldots) \tag{1-3}$$

where e_n is the vector of errors involving the current and several back points of integration, both for prediction and (any) correction; S_n is a matrix describing the propagation of error from one step to the next; and b_n describes the local truncation errors introduced at the current step. This Equation (1-3) has the general solution

$$e_n = E_n e_0 + \hat{e}_n \tag{1-4}$$

where e_0 is the initial starting error; $\hat{e}_n$ is the special error solution with zero initial error; and E_n is the matrix representing the propagation of an arbitrary initial error. E_n is given by

$$E_n = S_n S_{n-1} \cdots S_1 \tag{1-5}$$

[We note in passing, that $E_n e_0$ is the general solution of the homogeneous difference equation

$$e_n = E_n e_{n-1}, \qquad (n = 1,2,3,\ldots) \tag{1-6}$$

where e_0 is arbitrary. Roughly speaking, $E_n e_0$ is the error one would get at step n if there were no truncation error.]

Thus, by using the techniques sketched above, we are able to analyze the erros and determine their sources for a variety of multistep methods, step-sizes, algorithms (prediction, corrections, partial evaluation), and differential-equation formulations.

Algorithms, formulations and algorithm codes

The Algorithm/Formulation combinations which can be studied with the Analyzer are as follows:

Code 1 (PE): Denotes integration by prediction only, followed of course, by evaluation of the function f, where the equation being integrated has the form

$$y^{(j)} = f(t,y) \ . \tag{1-7}$$

Code 2 (PEC): Denotes integration using one correction after each prediction and evaluation, using the Formulation (1-7) above.

Code 4 (PECE*): Denotes PEC above, followed by a partial evaluation, that is, evaluation of the two-body Keplerian acceleration, without re-evaluation of the perturbation component in acceleration. This case assumes that the system of Equations (1-7) is the system of equations of motion in the Cowell formulation. That is, y denotes position in inertial Cartesian coordinates and $f(t,y)$ denotes total acceleration with respect to these coordinates.

Code 5 (Class I Special Formulation): Denotes integration of the first-order equations

$$\begin{aligned} y' &= v \\ v' &= f(t,y,v) \end{aligned} \qquad (1\text{-}8)$$

in the PECE* made. The same coordinate and acceleration definitions apply as for Code 4. Here, v denotes the velocity in this coordinate system.

Code 6 (Class I/II Special Formulation): Denotes integration of the mixed first-order and second-order system of equations:

$$\begin{aligned} y^{(2)} &= f(t,y,v) \\ v' &= f(t,y,v) \end{aligned} \qquad (1\text{-}9)$$

using a Class II method for the first part and a Class I method for the second part. These are integrated in the PECE* mode. Again, y denotes position in inertial Cartesian coordinates, and f and v denote acceleration and velocity, respectively, with respect to the inertial Cartesian coordinates system.

Input

I. Card Input

Card input for the Analyzer is required in the following form and sequence:

Items	Format
1. Algorithm code, class code, number of equations	3I5

The algorithm code is an integer from 1 to 8, representing algorithms of method combinations and/or basic formulations of differential equations. The codes are given by the following list (details are explained later):

Code	Algorithm
1	PE
2	PEC
3	Reserved for PECE
4	PECE*
5	Class I special Cowell formulation
6	Class I/II special Cowell formulation
7	Reserved for VOP
8	Reserved for K-S regularization

The class code is 1 or 2 for algorithms 1, 2 and 3, indicating the class of differential equations for integration and the class of the numerical method used.

The Class code is ignored for the algorithm codes 4, 5, 6, 7 and 8. The "number of equations" is the number of differential equations to be integrated. The input value of the "number of equations" is ignored for algorithm codes 4, 5 and 6.

Next Item	Format
2. Step number	I5
3. List of predictor a's (k +1 values, punched on as many cards as necessary)	3D25.0
4. List of predictor b's (as above)	3D25.0
5. List of corrector a's (as above)	3D25.0
6. List of corrector b's (as above)	3D25.0
7. List of Class I predictor a's (as above) Items 7, 8, 9 and 10 apply only to algorithm code 6 (Class I/II Special Cowell formulation), in which case, items 3, 4, 5 and 6 are the Class II coefficients. Otherwise, items 7 through 10 are omitted.	3D25.0
8. List of Class I predictor b's (as above)	3D25.0
9. List of Class I corrector a's (as above)	3D25.0

10. List of Class I corrector b's (as above) 3D25.0

11. Desired step size for simulated integration D25.0

12. Spectrum period, norm period, plot period, basic print period, meximum number of simulated steps 5I5

This concludes the card input.

The spectrum period is the number of simulated steps per computation of the spectrum of S_n and E_n. The norm period is the number of simulated steps per computation of the natural operator norm of E_n corresponding to the Euclidean vector norm. The "maximum number of simulated steps" over-rides the full use of the tape, provided this input value is greater than zero and less than the number allowed by the full use of the tape.

If the "spectrum period" is zero, the spectrum is not computed. If the "norm period" is zero, the norm E_n is not computed. If the "plot period" is zero, no plot is generated. And, if the "basic print period" is zero (or blank), it is reset to 1.

II. <u>Tape Input</u>

Binary tape input (file 1) for the Analyzer has the form:

Record 1: 100 integer words of character-string information (5 cards worth) used as a label for identifying information.

Record 2: Tape step-size (double precision), number of tape steps (integer)

The tape step-size is the step-size for generating the tape. The number of tape steps is the total number of steps (records) written on the tape (following this record).

Remaining records (all double precision):

a) Type 1 format (for algorithms 1, 2 and 4):

NEQ ordinate components, y_n; NEQ first or second-derivative components, y'_n or y''_n (depending on Class); the NEQ by NEQ Jacobian matrix

$$\frac{\partial F}{\partial y} (t_n, y_n)$$

where $y^{(j)} = f(t,y)$ and where NEQ is the number of scalar equations being integrated as a system. (NEQ is assumed = 3 for algorithm 4.)

b) Type 2 format (for algorithms 5 and 6):

3 ordinate components, y_n; 3 derivative components, y'_n; 3 second-derivative components, y''_n, followed by the extended Jacobian matrix

$$\frac{\partial F}{\partial [y, \dot{y}]} (t_n, y_n, \dot{y}_n)$$

where $y^{(2)} = f(t,y,\dot{y})$. The row indices of the Jacobian matrix are associated with the components of f, whereas, the components of y, then $\dot{y}$. The Jacobian matrix is assumed to be written on tape, one column at a time.

Output

Output for the Analyzer includes the norm of b_n, the natural norm of E_n (using the Eculidean vector norm), the spectral radius of S_n (computed only when the spectrum of S_n is computed), and the norm of the first three components of $e_n^{(0)}$, the special solution with zero initial error. Output also includes the norms of T^*, S^*, T^{**} and S^{**}, assuming three components for each. Quantities not computed are printed as zero. Card input item 12 defines print controls (see section on card input).

Details of graphed output are given later in this section, under the heading "Numerical Results of Multistep Analyzer."

Derivation of Error Equations

In this section, we sketch the derivation of the error equations of the form (1-3), discussed in the introduction.

Let $y_n = y(t_n)$, $t_n = t_0 + nh$, where h is the step-size of integration and y is the true solution of the differential equation

$$y^{(j)} = f(t,y,\dot{y}) \ , \tag{1-10}$$

where f may, or may not depend on $\dot{y}$.

Also, let $y_n^{(j)} = y^{(j)}(t_n)$. We define the truncation error for the predictor and corrector, respectively:

$$T_n^* = \sum_{i=0}^{k} a_i^* \, y_{n-1} + h^j \sum_{i=1}^{k} b_i^* \, y_{n-i}^{(j)} \tag{1-11}$$

$$T_n^{**} = \sum_{i=0}^{k} a_i \, y_{n-i} + h^j \sum_{i=0}^{k} b_i \, y_{n-i}^{(j)} \ . \tag{1-12}$$

In the mixed Class I/Class II case, where $j = 2$, we also integrate simultaneously

$$v' = f(t,y,v) \tag{1-13}$$

and we define the Class I truncation errors for prediction and correction, respectively, as follows (c's and d's replace a's and b's to distinguish the Class I methods from the Class II methods):

$$S_n^* = \sum_{i=0}^{k} c_i^* \, y_{n-1}^{(1)} + h \sum_{i=1}^{k} d_i^* \, y_{n-1}^{(2)} \tag{1-14}$$

$$S_n^{**} = \sum_{i=0}^{k} c_i \, y_{n-i}^{(1)} + h \sum_{i=0}^{k} d_i \, y_{n-i}^{(2)} \; . \qquad (1\text{-}15)$$

Difference equations for actual numerical integration (assuming no round-off error) are given as follows, starting with the PEC case where y_n^p denotes predicted values and y_n^c denotes corrected values:

$$y_n^p = \sum_{i=1}^{k} a_i^* \, y_{n-i}^c + h^j \sum_{i=1}^{k} b_i^* \, y_{n-i}^{(j)p} \qquad (1\text{-}16)$$

$$y_n^c = \sum_{i=1}^{k} a_i \, y_{n-i}^c + h^j \sum_{i=0}^{k} b_i \, y_{n-i}^{(j)p} \qquad (1\text{-}17)$$

where $y_n^{(j)p}$ denotes the evaluation of f at (t_n, y_n^p). Using the definitions of T_n^* and T_n^{**} and $e_n^p = y_n^p - y_n$, $e_n^c = y_n^c - y_n$, we obtain

$$e_n^p = \sum_{i=1}^{k} a_i^* \, e_{n-i}^c + h^j \sum_{i=1}^{k} b_i^* \, \frac{\partial f}{\partial y} \, (t_{n-i}, y_{n-i}^*) e_{n-i}^p + T_n^* \qquad (1\text{-}18)$$

$$e_n^c = \sum_{i=0}^{k} a_i \, e_{n-i}^c + h^j \sum_{i=0}^{k} b_i \, \frac{\partial f}{\partial y} \, (t_{n-i}, y_{n-i}^*) e_{n-i}^p + T_n^{**}. \qquad (1\text{-}19)$$

The y^*'s are points close to the y's. In the Analyzer we use the Jacobian matrices $\partial f/\partial y$, evaluated at y_n instead of y_n^*. A similar consideration and replacement is made for velocity values in other cases when

f depends also on velocity (i.e., $y^{(1)}$ when $j = 2$).

Thus, for PEC we obtain the vector difference equation having the form (1-3):

$$\begin{bmatrix} e_n^c \\ \vdots \\ e_{n-k+1}^c \\ \cdots\cdots \\ e_n^p \\ \vdots \\ e_{n-k+1}^p \end{bmatrix} = \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] \begin{bmatrix} e_{n-1}^c \\ \vdots \\ e_{n-k}^c \\ \cdots \\ e_{n-1}^p \\ \vdots \\ e_{n-k}^p \end{bmatrix} + \begin{bmatrix} T_n^{**} + b_0 J_n T_n^* \\ 0 \\ \vdots \\ 0 \\ \cdots\cdots\cdots \\ T_n^* \\ 0 \\ \vdots \\ 0 \end{bmatrix} . \tag{1-20}$$

We note that e_n^c, e_n^p, T_n^{**} and T_n^* are column vectors whose number of components is equal to the number of differential equations integrated. Call this number NEQ. A, B, C, D of (1-20) are given by

$$A = \begin{bmatrix} a_1 I + b_0 J_n a_1^* & \cdots & a_k I + b_0 J_n a_k^* \\ I & & \\ & \ddots & \\ 0 & & I \quad 0 \end{bmatrix} .$$

The top entries above and the I's (identities) are submatrices of dimension NEQ $\times$ NEQ.

$$B = \begin{bmatrix} (b_1 I + b_0 J_n b_1^*) J_{n-1} & \cdots & (b_k I + b_0 J_n b_k^*) J_{n-k} \\ & 0 & \end{bmatrix}$$

$$C = \begin{bmatrix} a_1^* I & \cdots & a_k^* I \\ & 0 & \end{bmatrix}$$

$$D = \begin{bmatrix} b_1^* J_{n-1} & \cdots & b_k^* J_{n-k} & \\ I & & 0 & \\ & \ddots & & \\ 0 & & I & 0 \end{bmatrix}$$

where, of course, A, B, C, D are matrices of the same size and

$$J_n = \frac{\partial f}{\partial y} (t_n, y_n, \dot{y}_n) \ .$$

For simplicity, the b's and b*'s are assumed to be multiplied by h^j already.

Exactly similar considerations apply to the derivation of equations for the other algorithms. They are summarized here.

<u>For</u> PE:

$$
\begin{bmatrix} e_n^p \\ \vdots \\ e_{n-k+1}^p \end{bmatrix}
=
\begin{bmatrix}
a_1^* I + b_1^* J_{n-1} & \cdots & a_k^* I + b_k^* J_{n-k} & \\
I & & 0 & \\
 & \ddots & & \\
0 & & I & 0
\end{bmatrix}
\times
\begin{bmatrix} e_{n-1}^p \\ \vdots \\ e_{n-k}^p \end{bmatrix}
+
\begin{bmatrix} T_p^* \\ 0 \\ \vdots \\ 0 \end{bmatrix} .
\tag{1-21}
$$

<u>For</u> PECE* (E* = partial **evaluation**):

$$
\left[\begin{array}{c} e_n^c \\ \vdots \\ e_{n-k+1}^c \\ \hline e_n^p \\ \vdots \\ e_{n-k+1}^p \end{array}\right]
=
\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]
\left[\begin{array}{c} e_{n-1}^c \\ \vdots \\ e_{n-k}^c \\ \hline e_{n-1}^p \\ \vdots \\ e_{n-k}^p \end{array}\right]
+
\left[\begin{array}{c} T_n^{**} + b_0 J_n T_n^* \\ 0 \\ \vdots \\ 0 \\ \hline T_n^* \\ 0 \\ \vdots \\ 0 \end{array}\right]
\tag{1-22}
$$

where

$$
A = \begin{bmatrix}
a_1 I + b_1 K_{n-1} + b_0 J_n (a_1^* I + b_1^* K_{n-1}) & \cdots & a_k I + b_k K_{n-k} + b_0 J_n (a_k^* I + b_k^* K_{n-k}) \\
I & & 0 \\
 & \ddots & \\
0 & & I \quad \underbrace{\qquad 0 \qquad}_{\text{NEQ zero entries}}
\end{bmatrix}
$$

$$K_n = \frac{\partial f_k}{\partial y} (y_n) \qquad \text{where} \qquad f = f_K + f_P .$$

Here f_K is the Keplerian two-body acceleration and f_P is the perturbation in acceleration.

We note that the matrices K_n, $P_n = J_n - K_n$ are computed in the Analyzer using the values of y_n and J_n read from the tape input. K_n is given by

$$
K = \begin{bmatrix}
-\frac{\mu}{r^3}\left(1 - \frac{3}{r^2} y_1^2\right) & \frac{3\mu}{r^5} y_1 y_2 & \frac{3\mu}{r^5} y_1 y_3 \\
\frac{3\mu}{r^5} y_1 y_2 & -\frac{\mu}{r^3}\left(1 - \frac{3}{r^2} y_2^2\right) & \frac{3\mu}{r^5} y_2 y_3 \\
\frac{3\mu}{r^5} y_1 y_3 & \frac{3\mu}{r^5} y_2 y_3 & -\frac{\mu}{r^3}\left(1 - \frac{3}{r^2} y_2^2\right)
\end{bmatrix}
$$

where the index n has been dropped and $r = |y|$, and y_1, y_2 and y_3 are the three components of y (the position vector).

The remaining submatrices of S_n are given as follows:

$$B = \begin{bmatrix} b_1 P_{n-1} + b_0 J_n b_1^* P_{n-1} & \cdots & b_k P_{n-k} + b_0 J_n b_k^* P_{n-k} \\ & 0 & \end{bmatrix}$$

$$P_n = \frac{\partial f_p}{\partial y} (y_n)$$

$$C = \begin{bmatrix} a_1^* + b_1^* K_{n-1} & \cdots & a_k^* + b_k^* K_{n-k} \\ & 0 & \end{bmatrix}$$

$$D = \begin{bmatrix} b^* P_{n-1} & \cdots & & b_k^* P_{n-k} \\ I & & & \\ & \ddots & & 0 \\ 0 & & \ddots & \\ & & I & 0 \end{bmatrix} .$$

We note that for the Class I special formulation where the algorithm is PECE*, we write

$$\dot{y} = v$$

$$\dot{v} = f(t,y,v)$$

that is

$$\begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ f_K(y) \end{bmatrix} + \begin{bmatrix} 0 \\ f_P(y,v) \end{bmatrix} = \hat{f}_K + \hat{f}_P \; .$$

So that the new K's and P's (denoted by hats) are given as follows:

$$\hat{K}_n = \frac{\partial \hat{f}_K}{\partial [y,v]} (t_n, y_n, v_n) = \left[\begin{array}{c:c} 0 & I \\ \hdashline K_n & 0 \end{array} \right]$$

$$\hat{P}_n = \frac{\partial \hat{f}_P}{\partial [y,v]} = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline P_n & \frac{\partial f_P}{\partial v} (t_n, y_n, v_n) \end{array} \right]$$

and $\hat{n}_{eq} = 6$.

<u>For</u> PECE* <u>Class I/II special formulation</u>:

For this case we simultaneously integrate:

$$y^{(2)} = f(t,y,v)$$

by a Class II method and

$$v' = f(t,y,v)$$

by a Class I method. The error equation is given by the following (where $\dot{e}_n^p = v_n^p - y_n^{(1)}$, $\dot{e}_n^c = v_n^c - y_n^{(1)}$, and v_n^p and v_n^c are, respectively, the predicted and corrected values of v, the velocity):

$$\begin{bmatrix} e_n^c \\ \dot{e}_n^c \\ \vdots \\ e_{n-k+1}^c \\ \dot{e}_{n-k+1}^c \\ \hdashline e_n^p \\ \dot{e}_n^p \\ \vdots \\ e_{n-k+1}^p \\ \dot{e}_{n-k+1}^p \end{bmatrix} = \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] \begin{bmatrix} e_{n-1}^c \\ \dot{e}_{n-1}^c \\ \vdots \\ e_{n-k}^c \\ \dot{e}_{n-k}^c \\ \hdashline e_{n-1}^p \\ \dot{e}_{n-1}^p \\ \vdots \\ e_{n-k}^p \\ \dot{e}_{n-k}^p \end{bmatrix}$$

$$+ \begin{bmatrix} \begin{bmatrix} T_n^{**} \\ S_n^{**} \end{bmatrix} + \begin{bmatrix} b_0 J_n \\ d_0 J_n \end{bmatrix} \begin{bmatrix} T_n^{*} \\ S_n^{*} \end{bmatrix} \\ 0 \\ \vdots \\ 0 \\ \hdashline T_n^{*} \\ S_n^{*} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where

$$
A = \begin{bmatrix}
\left[\begin{array}{c:c} a_1 I + b_1 K_{n-1} & 0 \\ \hdashline d_1 K_{n-1} & c_1 I \end{array}\right] + \begin{bmatrix} b_0 J_n \\ d_0 J_n \end{bmatrix} \left[\begin{array}{c:c} a_1^* I + b_1^* K_{n-1} & 0 \\ \hdashline d_1^* K_{n-1} & c_1^* I \end{array}\right] & \cdots & & \\
I_{\hat{n}_{eq}} & & 0 & \\
 & \ddots & & \\
0 & & I_{\hat{n}_{eq}} & \underbrace{0}_{\hat{n}_{eq} \text{ zeros}}
\end{bmatrix}
$$

Dots on the top row indicate entries like the upper-left submatrix of A shown, except that the index 1 is replaced by an index running up to k for subsequent submatrices. We note that, for simplicity of expression, the b's and b^*'s have been multiplied by h^2, whereas the d's and d^*'s have been multiplied by h.

$$
B = \begin{bmatrix}
\begin{bmatrix} b_1 P_{n-1} \\ d_1 P_{n-1} \end{bmatrix} + \begin{bmatrix} b_0 J_n \\ d_0 J_n \end{bmatrix} \begin{bmatrix} b_1^* P_{n-1} \\ d_1^* P_{n-1} \end{bmatrix} & \cdots \\
 & \\
0 &
\end{bmatrix}
$$

$$C = \begin{bmatrix} \begin{bmatrix} a_1^* I + b_1^* K_{n-1} & \vdots & 0 \\ \cdots\cdots\cdots & \cdot & \cdots \\ d_1^* K_{n-1} & \vdots & c_1^* I \end{bmatrix} & \cdots \\ & \\ 0 & \end{bmatrix}$$

$$D = \begin{bmatrix} \begin{bmatrix} b_1^* P_{n-1} \\ d_1^* P_{n-1} \end{bmatrix} & \cdots & \\ I & & \\ & \ddots & \\ & & I \quad 0 \end{bmatrix}.$$

The upper dots in B, C and D have the exactly similar meaning as for A.

1.3 Numerical Results of General Multistep Analyzer

This section is designed to exhibit the capabilities of the Analyzer program. Four trajectories have been used in computations with the Analyzer.

Trajectory 1 is a circular trajectory with a two hour period. (The semi-major axis is approximately 8,059 kms., $i = 45^{\circ}$.) Trajectory 2 is

a similar Keplerian orbit, with a period of two hours, but with $e = .01$. Initial conditions are omitted, as they are not relevant. Trajectory 3 roughly simulates, in shape, and AEC trajectory: $a \sim 8,544$ kms., $e \sim .24$, and $i = 45^{\circ}$, with drag characteristics similar to the GEOSB vehicle. Geopotential coefficients are applied through 15^{th} degree zonals and 15 order tesserals. Solar and linear gravitational effects are included. Drag and solar radiation accelerations are defined by

Drag Coefficient	=	2.3
Satellite cross sections are	=	1.23 meters^2
Solar indication pressure	=	4.5×10^{-6} newtons per meter^2
Reflectivity	=	1.5

Ephemeris requested from 670510. In the related computations, perigree occurs at approximately 1630 secs. Trajectory 4 is the same as Trajectory 1 with J_2 effects and solar and lunar gravitation applied.

Figure 1-1 shows the norm of E_n (the composite error operator) through one revolution, using Trajectories 1 and 2. Here, $\|E_n\|^2$ = spectral radius $(E_n^* E_n)$ = square of maximum scaling of any real vector in the unit ball of the relevant space. The combined algorithms of Cowell Class II/ Adams/PE are used with step-size $h = 100$ secs. and step number k of the integrator = 11. The graph shows that the norm is quite independent of the change in eccentricity.

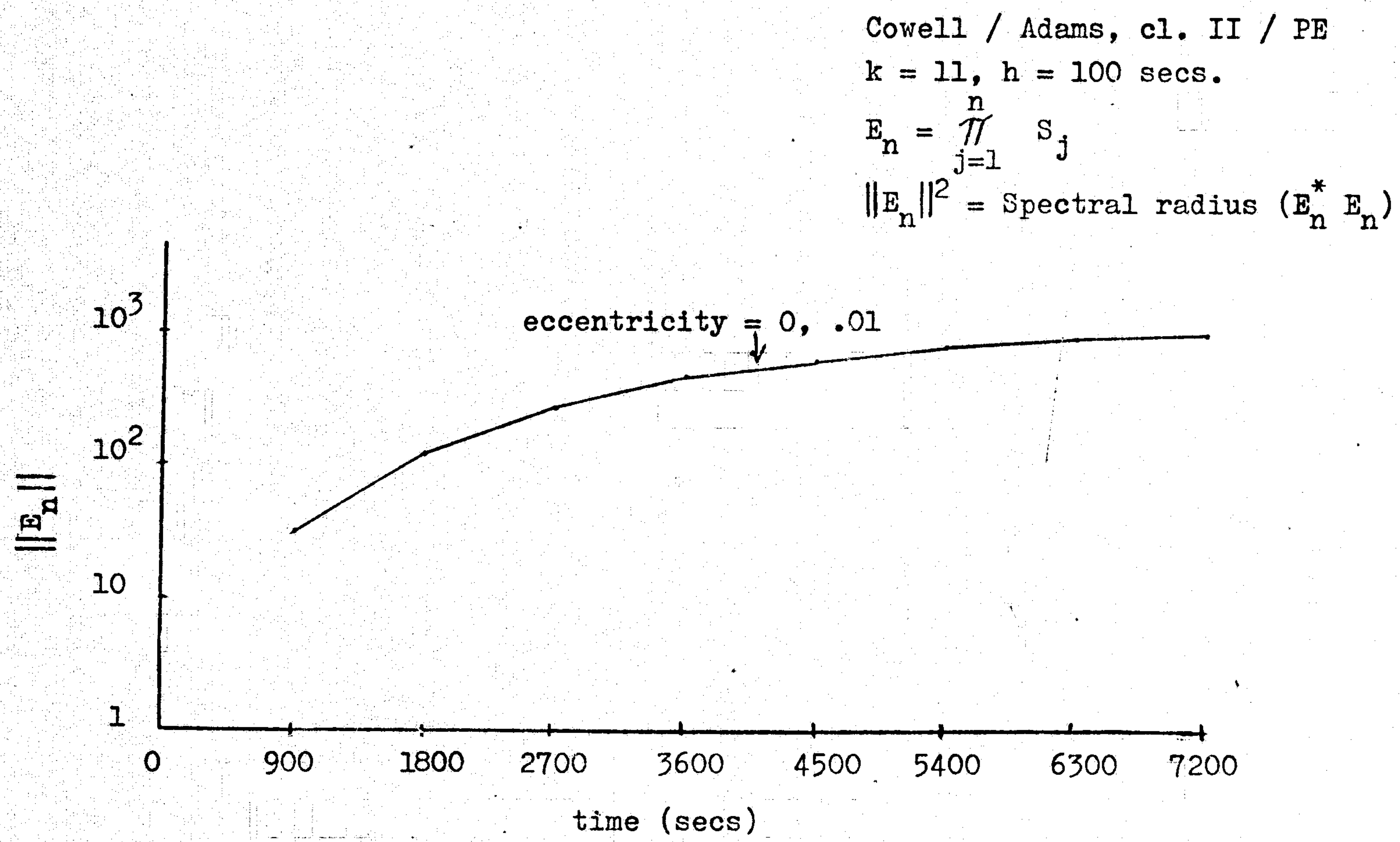


Figure 1-1 COMPARISON $\|E_n\|$, TRAJECTORIES 1 AND 2

Figure 1-2 is the result of computation of the spectral radius of the composite error operator E_n through approximately one revoluation of Trajectory 4. The same formulation -- integrator - integrator algorithm -- is used here, as in Figure 1. This figure would remain essentially unchanged if perturbations were omitted. The behavior of the spectral radius is radically different from the norm of Figure 1. This behavior of the spectral radius suggested use of the norm described above for error growth measurement. The integration is numerically stable in the sense that, locally, the non-principal roots (of S_n) are well within the unit circle. These eigenvalues are quite independent of position (of n), because of the circularity of the trajectory. Because of the symmetry of the Jacobian matrix (Keplerian), it is apparent that E_{72} is approximately equal to E_{36}^2. Equality holds on the unperturbed trajectory. Little can be inferred about the norm behavior if the computation were extended over several revolutions, except at the midpoint and the end of the cycle.

The spectral radius of E_{72}, without perturbations and with an exact two-hour period, is greater than one, because of a principal root. This indicates both non-numerical instability and an asymptotic error growth which is exponential. This is discussed in the subsection on asymptotic error.

Figure 1-3 compares local truncation error levels in computations of Trajectories 1 and 2, with the same integration configuration as Figure 1-1: Cowell, Class II/Adams/PE, and with the same step-number and

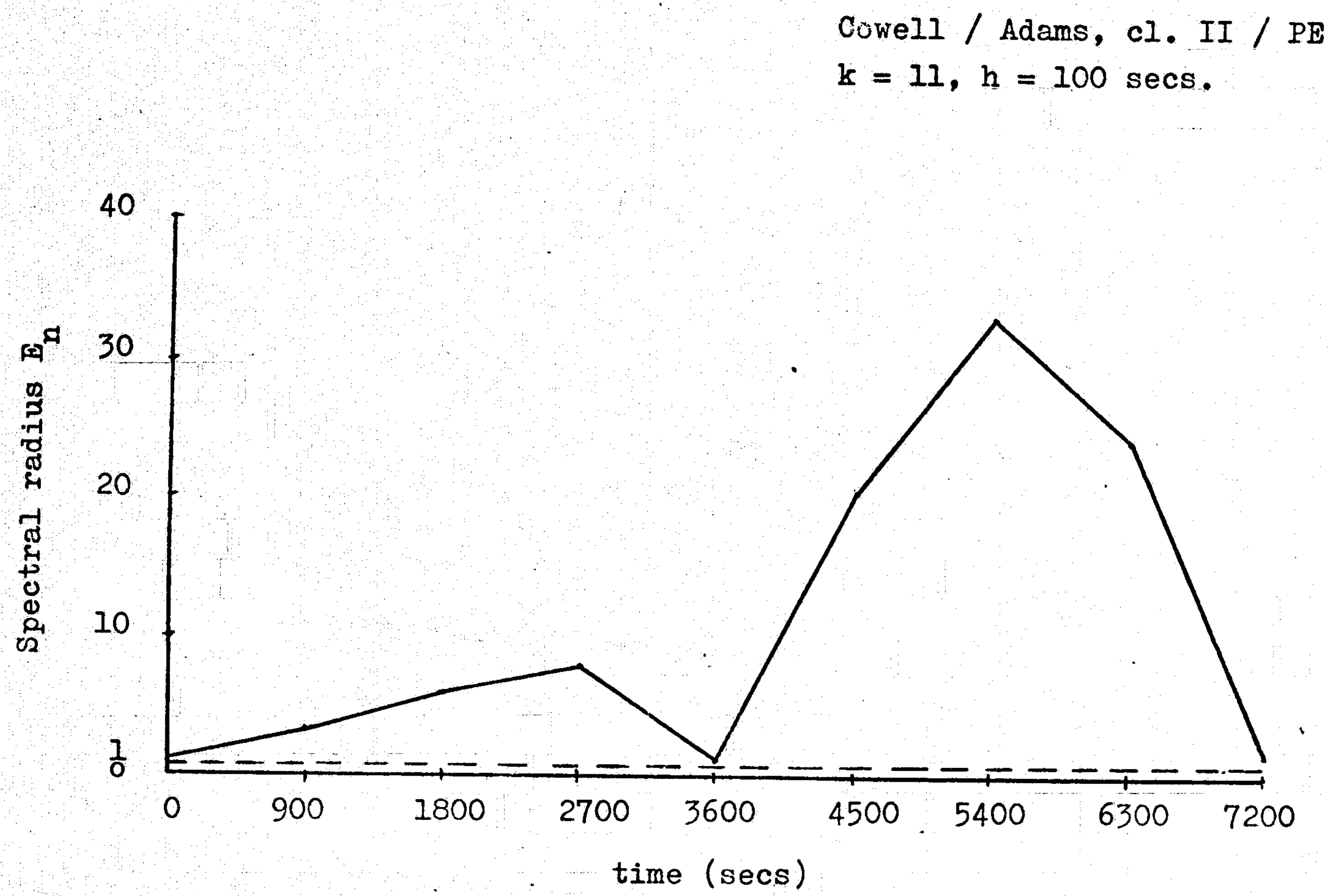


Figure 1-2 SPECTRAL RADIUS OF ERROR OPERATOR $E_n = \prod_{i=1}^{n} S_i$

TRAJECTORY 4

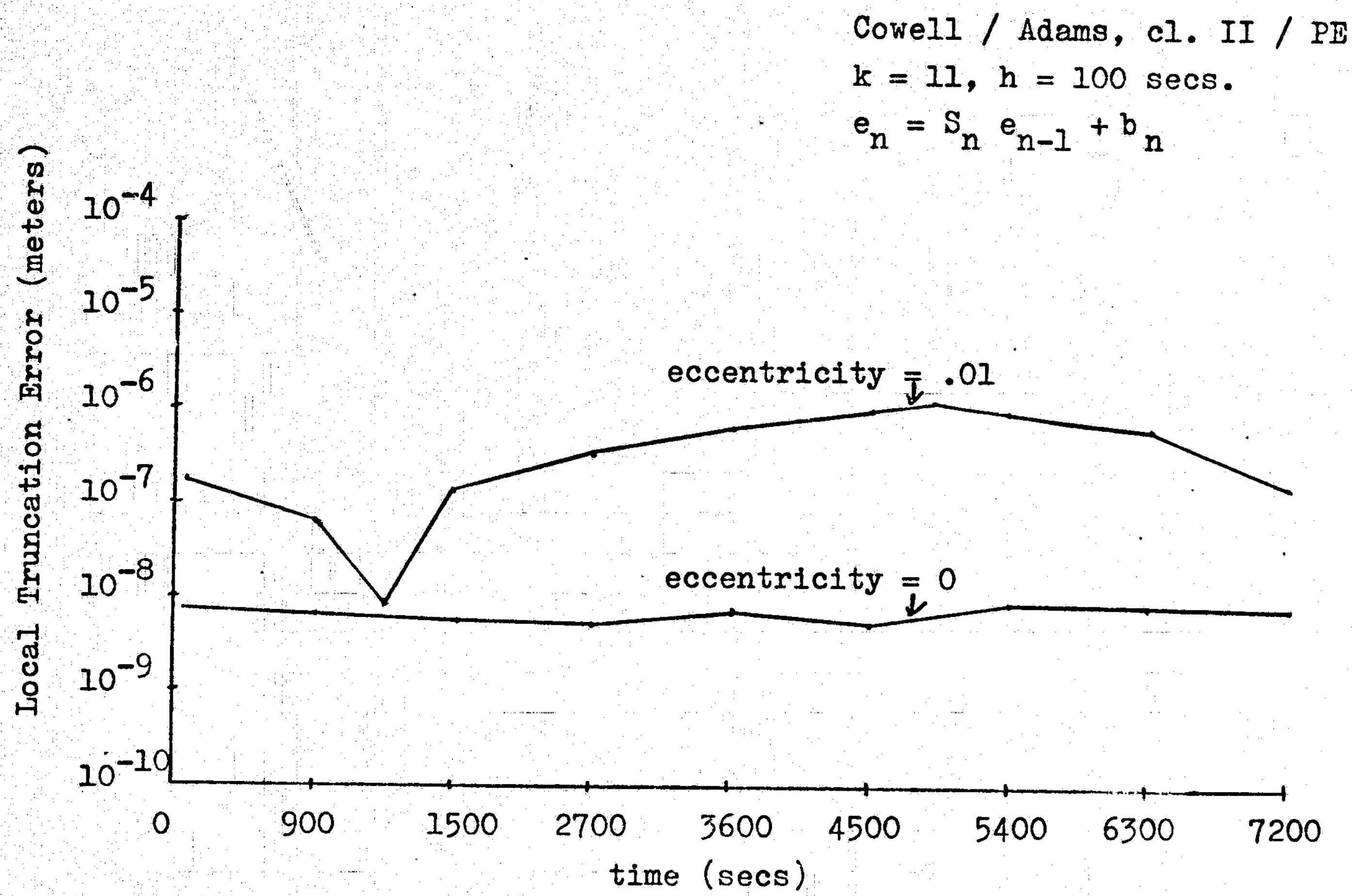


Figure 1-3 COMPARISON OF LOCAL TRUNCATION ERROR, $\|b_n\|$
TRAJECTORY 1 AND 2

step-size. It is seen that the local truncation error, in this case, depends heavily on the eccentricity. The vector norm used throughout is the usual Euclidean length (the operator norm defined above is the natural one for this vector norm).

In Figure 1-4, the propagated position error (partial norm of propagated error vector) is compared in the computation of Trajectories 1 and 2 with the same computational framework. The initial error vector $e_0 = 0$, throughout this report. In this case, the difference in the levels of propagated error, corresponding to the two eccentricities, is roughly equal to the difference in the average levels of local truncation error. Moreover, a reasonable estimate of propagated error, in this case, can be obtained as a product of the average level of local error and the norm of the error operator. The observations are made: Errors committed early in the computation are most important in the final propagated error. The truncation error level is relatively constant. Thus, early errors reflect this general level of local error. As time increases, only the composite error operator and the direction and magnitude of the early established error vector are important in the propagation. This method of crude approximation apparently is not valid in Class I integration, where the norm is characteristically much larger (see Figure 1-6).

Figure 1-5 shows propagated error and local truncation error in the computation of Trajectory 2 by Cowell, Class II/Adams, using integrators with step-number $k = 4$ and step-size $h = 100$ secs. The integrator algorithm PE is compared with algorithm PECE* = PECE.

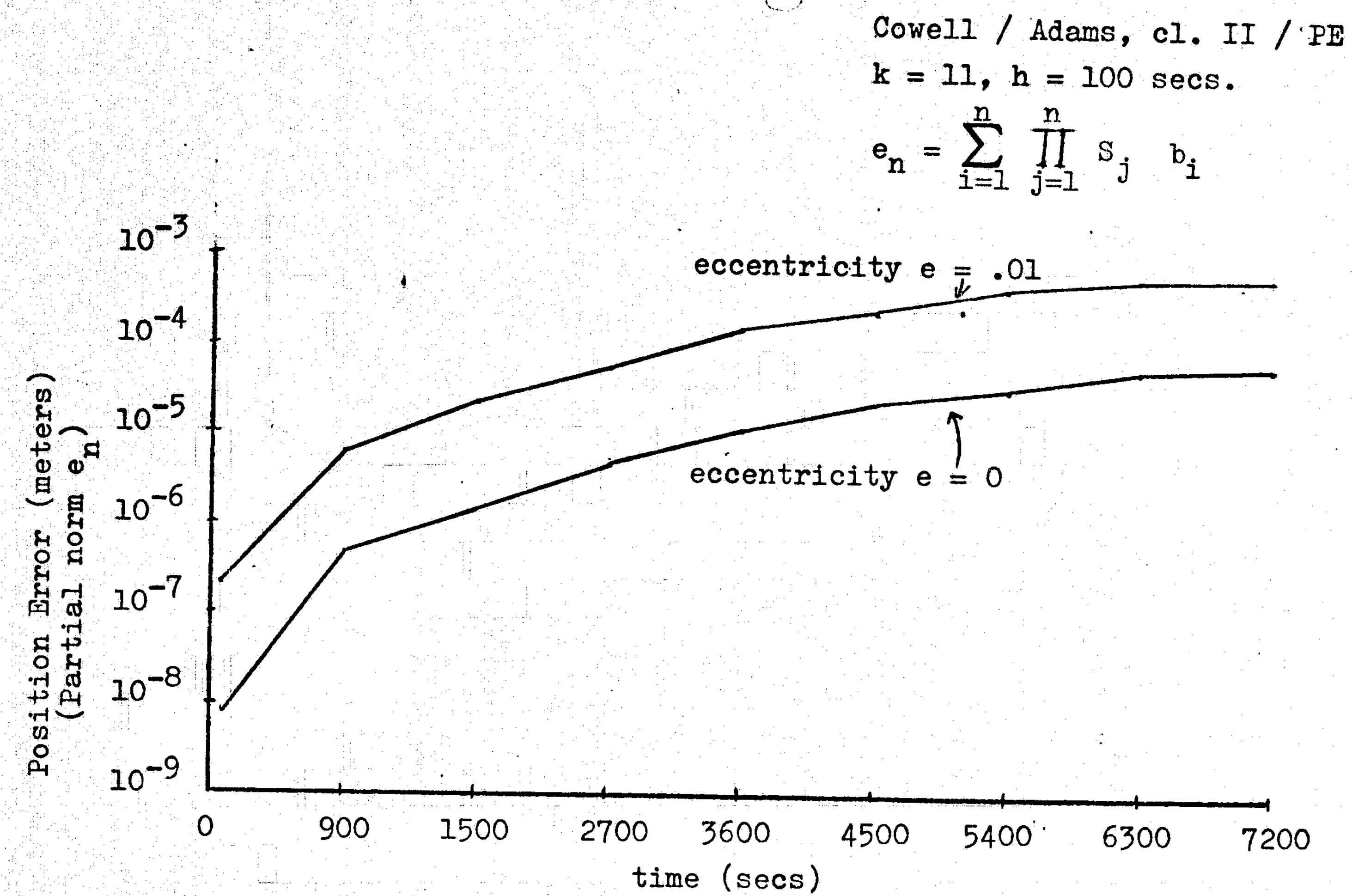


Figure 1-4 COMPARISON OF PROPAGATED POSITION ERROR
TRAJECTORIES 1 AND 2

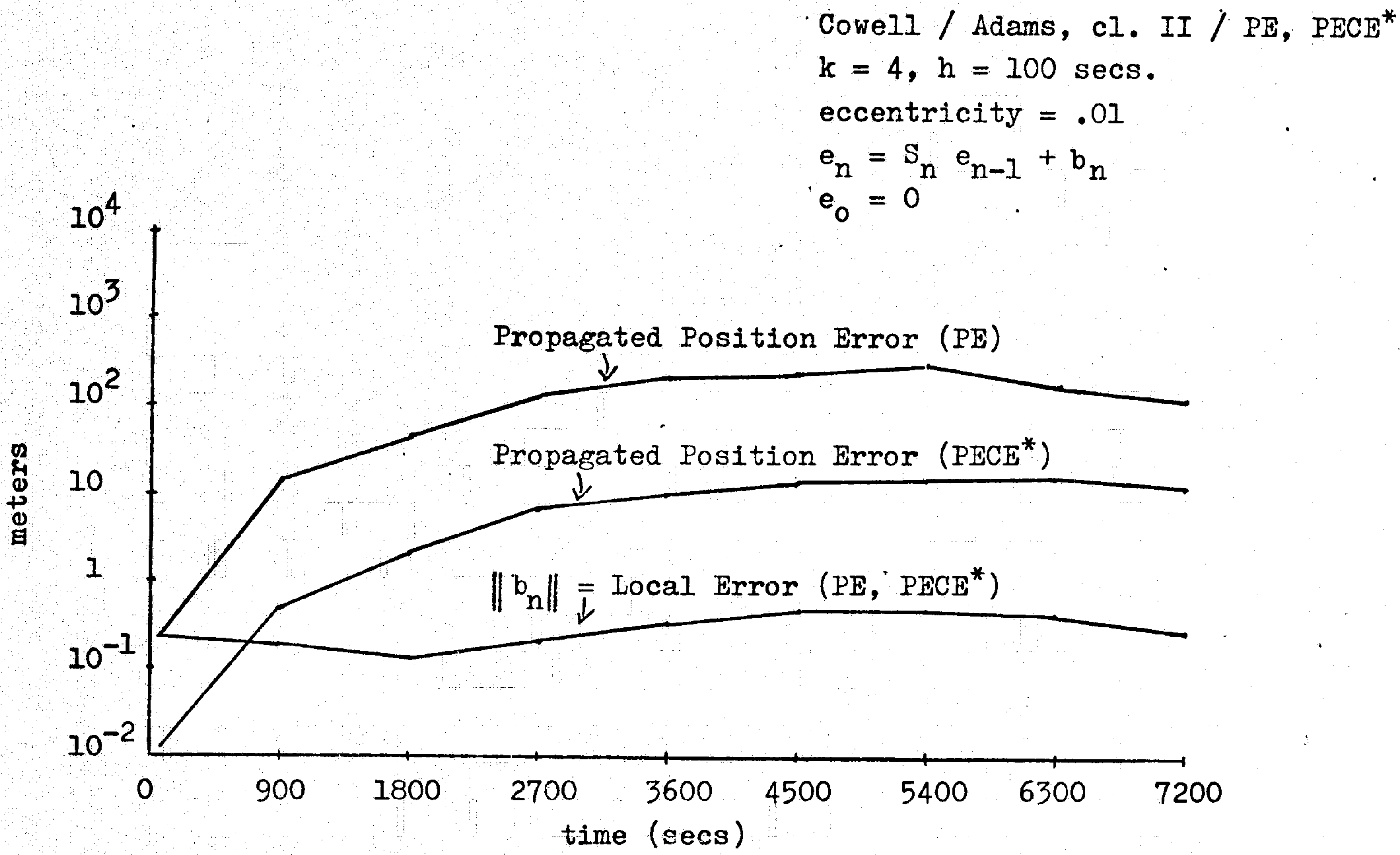


Figure 1-5 COMPARISON LOCAL ERROR, PROPAGATED ERROR TRAJECTORIES 1 AND 2

The norms (not shown) of the two-operators, as functions of time, are:

t (secs)	Approximate norm $\|E_n\|$ PE	PECE* = PECE
500	38	27
1400	117	83
2300	265	183
3200	471	332
4100	680	480
5000	827	584
5900	880	620

It must be pointed out that the local error vector, b_n, (for algorithm PECE*), has components equal to the local predictor error and the local predictor-corrector error for each coordinate and for every time point in the difference equation. The predictor error apparently dominated, here, yielding approximately the same local error vector norms for both PE and PECE*. Although it is as yet unverified, the conjecture is offered that the propagated error is roughly approximated by the product of the average level of <u>predictor-corrector error</u> (a partial norm of vector, b_n) and the norm of the error operator.

Figure 1-6 shows local and propagated error in the computation of Trajectory 2 with Cowell, Class I/Adams/PECE* = PECE configuration, with $k = 4$ and $h = 100$. The step-size used here is large for any except theoretical study. Error propagation is somewhat worse than the predictor-

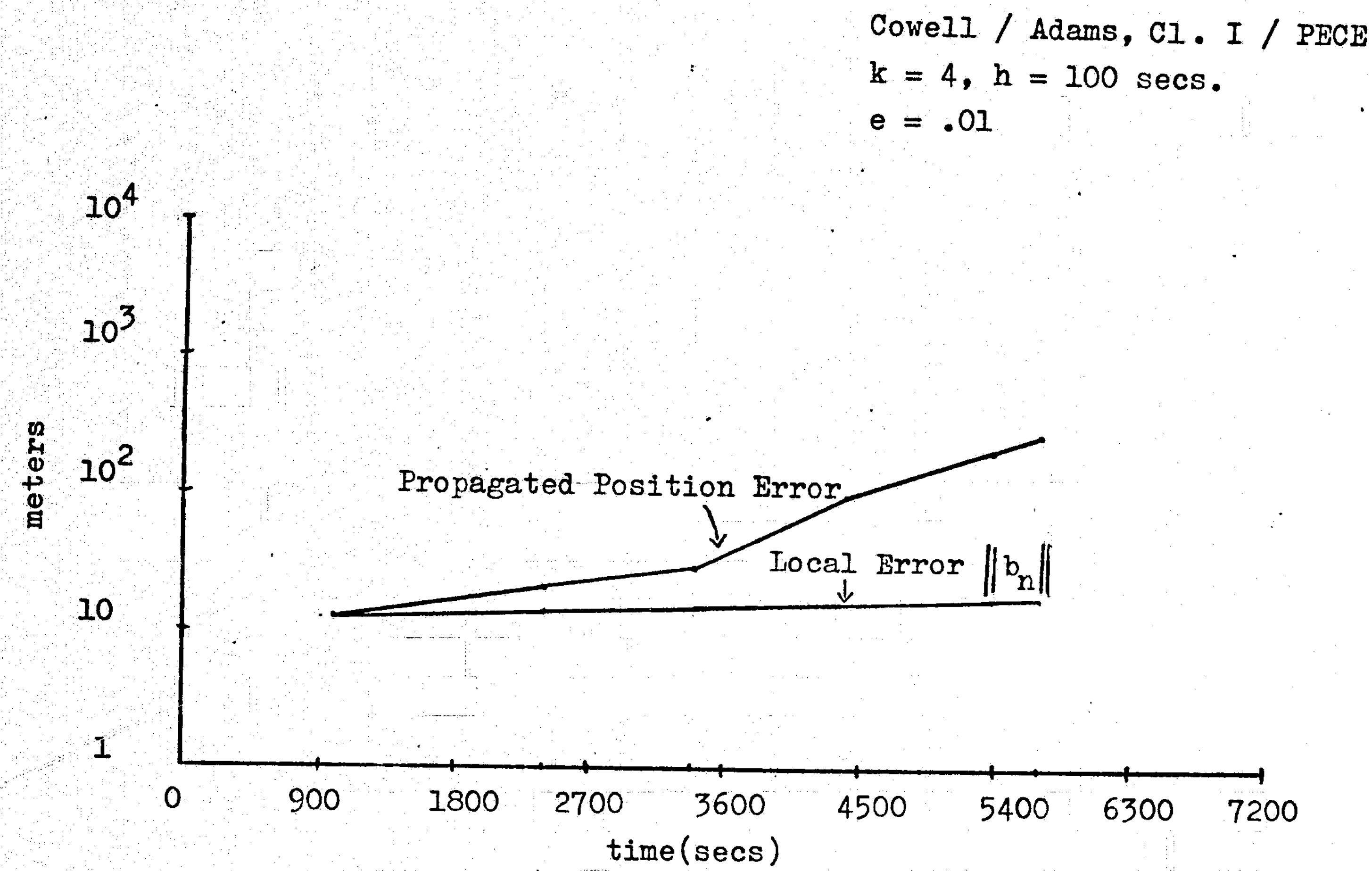


Figure 1-6 ERROR IN CLASS I INTEGRATION, TRAJECTORY 2.

only error in the previous figure. It is again point out that the local error vector has as components the pure predictor error as well as predictor-corrector error.

In Figure 1-7 we compared the norms of the error operator, E_n, for Trajectory 2 with Class I and Class II integration. The Class I norm exceeds the Class II norm by a factor of roughly 100. The factor may be exactly 100 and differences due to inaccuracies in the norm computation, although the reason for this is not clear. The tabulated values are:

	Approximate norm $\|E_n\|$	
t (secs)	Class I	Class II
900	2535	27
1800	8039	83
2700	18363	183
3600	32865	332
4500	47840	480

Figure 1-8 is the result of the printer plot subroutine of the Analyzer Program. Parameters in the computational configuration are indicated on the plot. The time scale is on the ordinate axis. The 1-curve is the norm of the composite error operator $\|E_n\|$. The 2-curve displays the propagated position error vector norm and the 3-curve is the local error vector norm (total norm). It is important to note that the three functions are individually normalized by dividing by their maximum values in the indicated domain. The log scale, used in previous figures, is not employed in the printer-plot routine.

Cowell / Adams, Cl.I and Cl.II/ PECE
$k = 4$, $h = 100$ secs.
$e = .01$

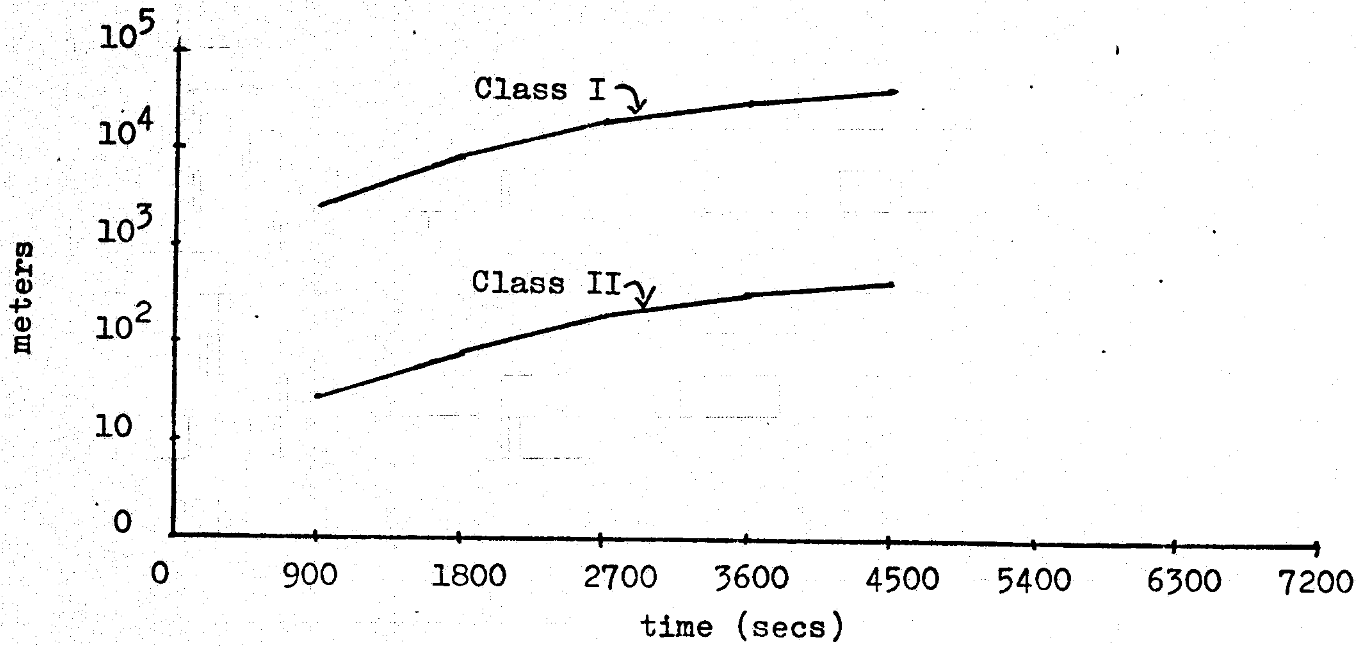


Figure 1-7 COMPARISON OF $\|E_n\|$, CL. I AND CL. II INTEGRATION TRAJECTORY 2

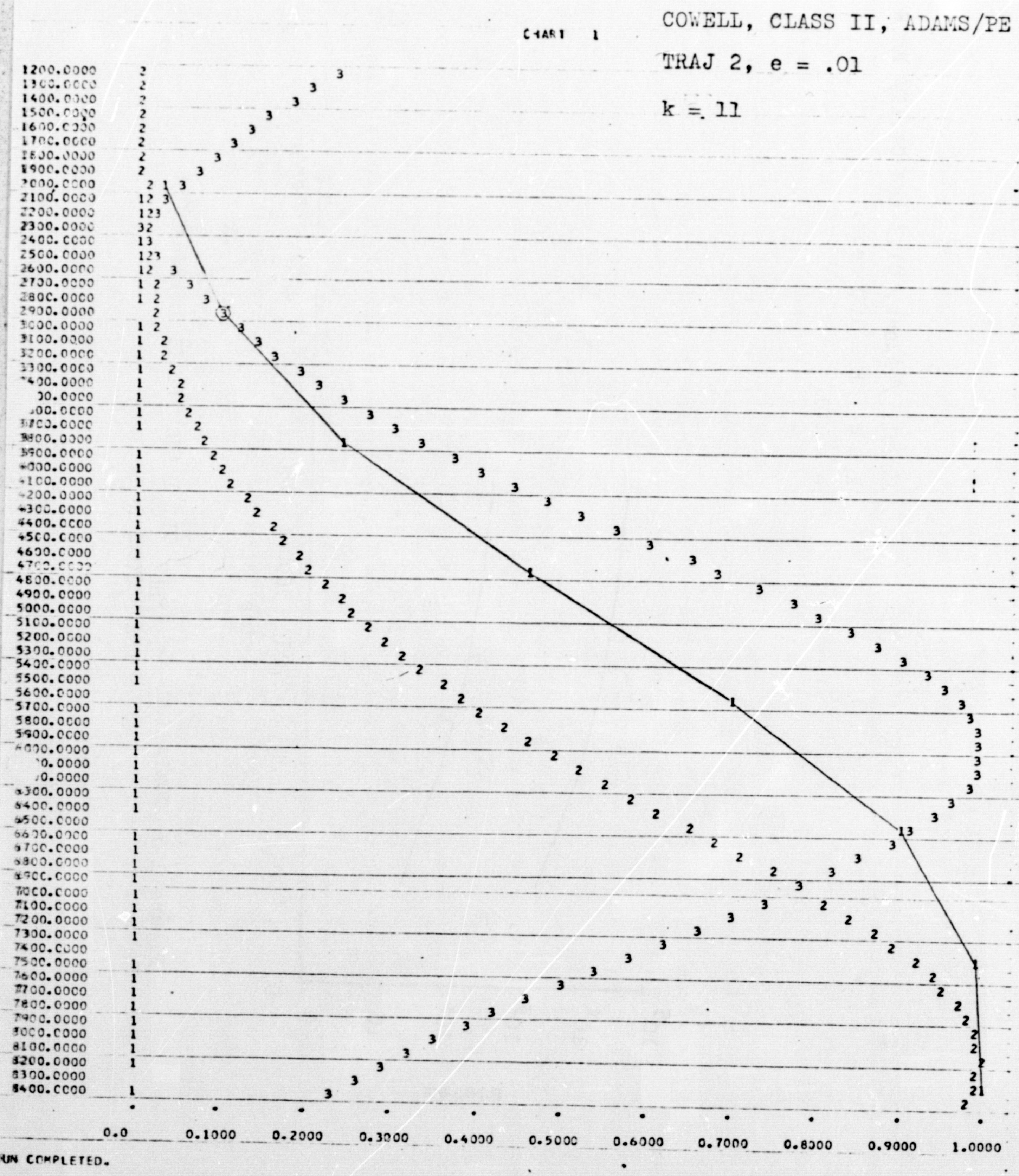


FIG. 1-8 ANALYZER PRINTER PLOT, TRAJ. 2.

The approximate individual normalization factors by which graphed functions are to be multiplied are, approximately:

Curve	Factor
1	$.101 \times 10^{4}$
2	$.646 \times 10^{-3}$
3	$.100 \times 10^{-3}$

The results of this integration are shown in different forms in Figures 1-1, 1-3 and 1-4. The computation of $\|E_n\|$ is done infrequently, to reduce computing expense. The frequency is optional in the program.

The remaining figures in this section are concerned with application of the Analyzer to a partially simulated AEC trajectory (Trajectory 3). The Cowell, Class I and Class II equations are studied with Adams integrators of step number 4 and 11. Integrator algorithm PECE* is used throughout. The step-size, although capable of variation by simple change of an input parameter, was fixed throughout this experimental analysis at 100 secs. Graphed results include the effect of perturbative accelerations given above in the trajectory description, with the exception of the first figure (1-9), where no drag was applied. Small arcs are studied, up to 3900 secs., which is about one-half a cycle. As is seen from the first two graphs, perigreee occurs at about 1630 seconds.

Figure 1-9 is the printer-plot output of a Class I/Class II integration, $k = 4$, of the simulated AEC trajectory without drag. The curves 1, 2 and 3 have the meaning defined in the description of Figure 1-8. The 4-curve is the predictor error vector norm and the 5-curve is the predictor-corrector error vector norm. The vector norms represented by plots 4 and 5 are partial norms of the total local error vector b_n, whose norm is the 3-plot. The pure corrector error does not appear explicitly as components of the local error vector.

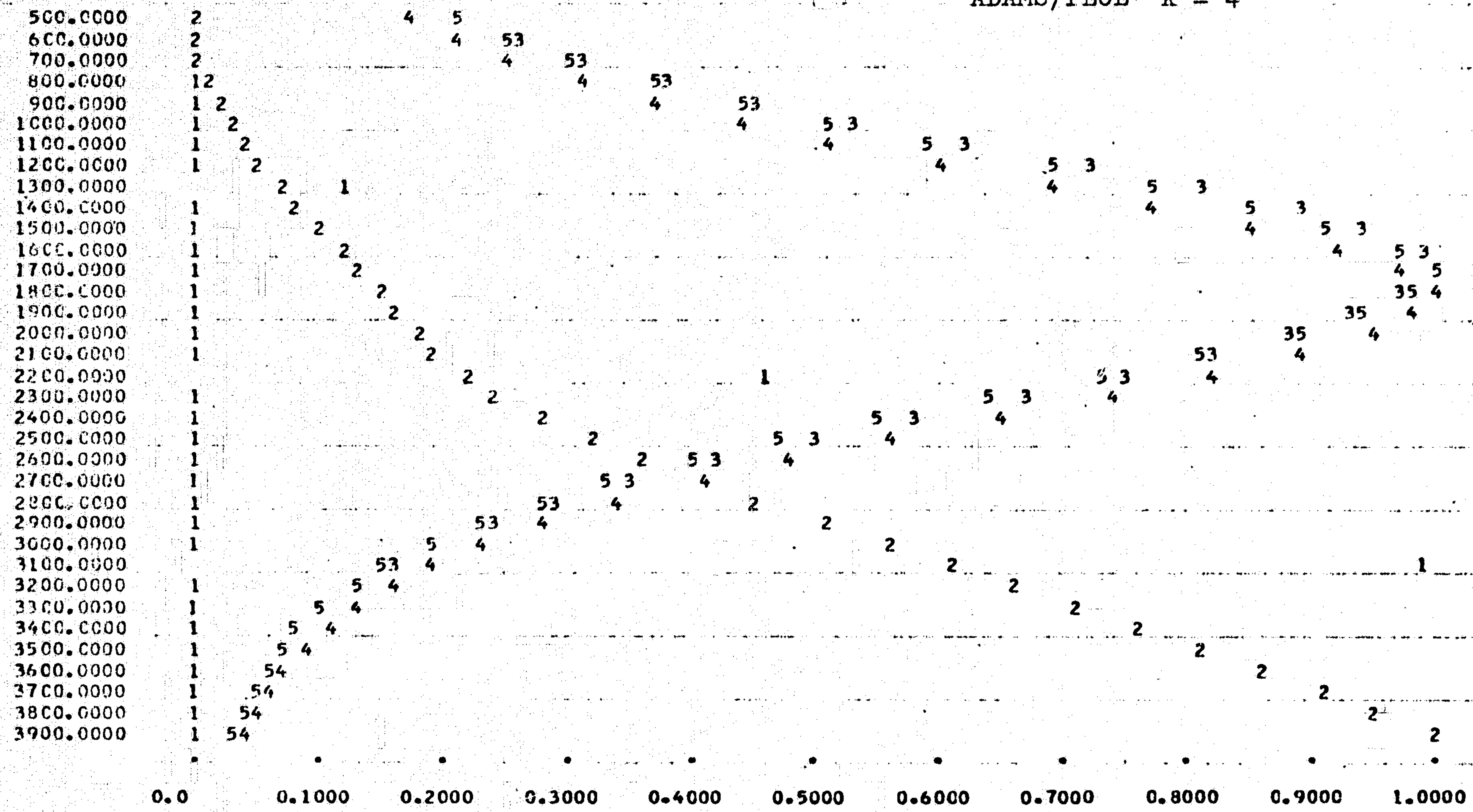
The normalization factors associated with numbered curves (by which graphed functions are to be multiplied) are approximately:

1.	307
2.	147
3.	15
4.	15

The 5^{th} scale factor does not appear directly as printed output of the Analyzer program.

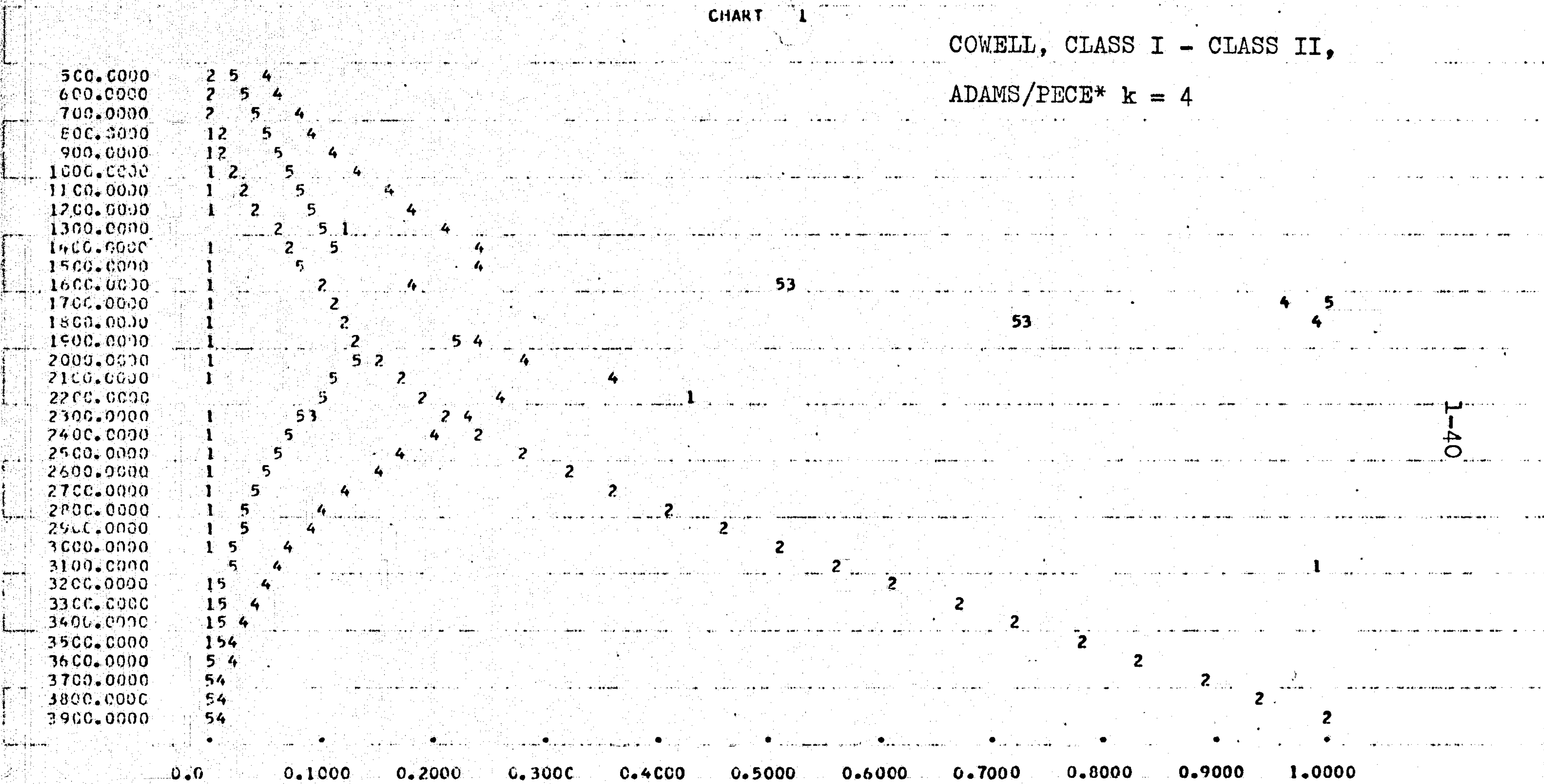
Figure 1-10 is the same AEC simulated computation as Figure 1-9, except that all above described forces are applied. Normalization factors are approximately:

CHART 1 COWELL, CLASS I – CLASS II,
ADAMS/PECE* k = 4



RUN COMPLETED.

Figure 1-9 ANALYZER PRINTER-PLOT, TRAJ. 3, NO DRAG



RUN COMPLETED.

Figure 1-10 ANALYZER PRINTER-PLOT, TRAJ. 3, DRAG APPLIED

1.	328
2.	182
3.	47
4.	47

The 5^{th} scale factor is not given. The remark given with reference to Figure 1-9 applies here.

Corresponding printer plots for remaining computational configuration are omitted. Rather, pertinent information is presented on semi-log plots.

Figure 1-11 compares propagated position error in the Cowell, Class I and Class I/Class II computation of the AEC trajectory, with full forces with algorithm PECE* and Adams integration of step-number $k = 4$ and step-size $h = 100$ secs.

Figure 1-12 presents the same material on a short arc and for $k = 11$.

The Euclidean norms of the total local error vectors for step-number $k = 4$ are compared for Class I and Class I/Class II integration in Figure 1-13, while Figure 1-14 gives a comparison of these quantities with step-number $k = 11$.

Cowell / Adams / PECE*

k = 4, h = 100 secs.

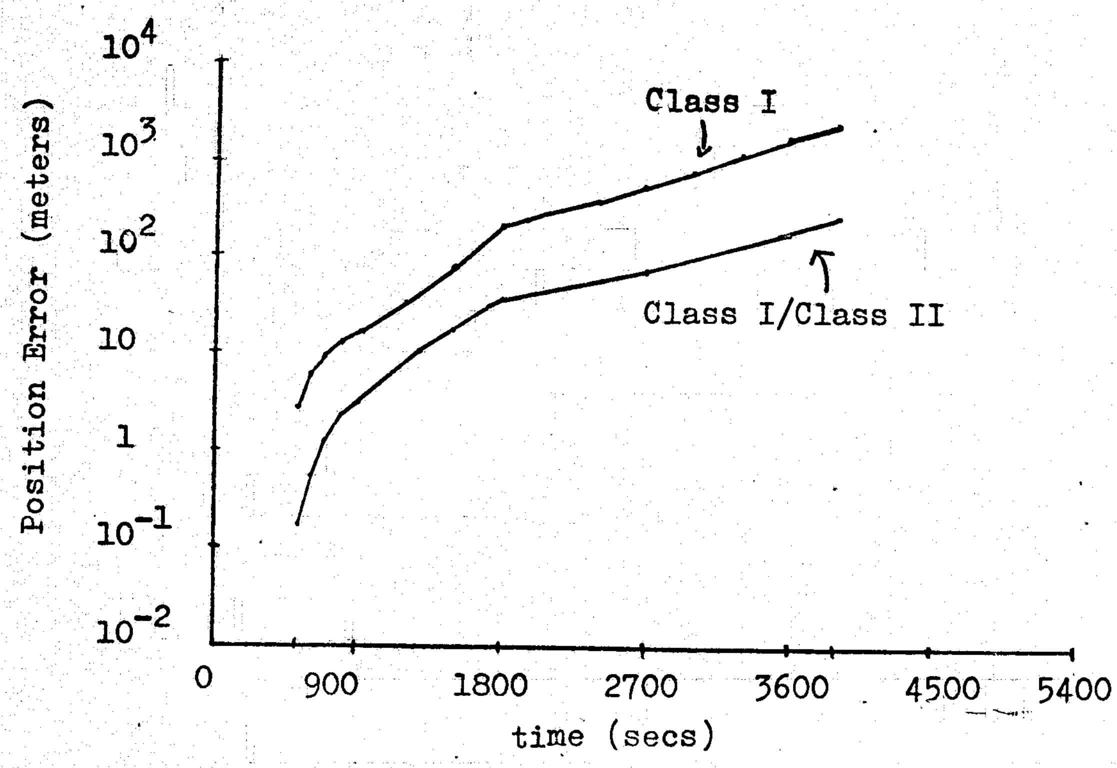


Figure 1-11 COMPARISON OF PROPAGATED ERROR IN CLASS I AND CLASS I/CLASS II INTEGRATION, TRAJECTORY 3.

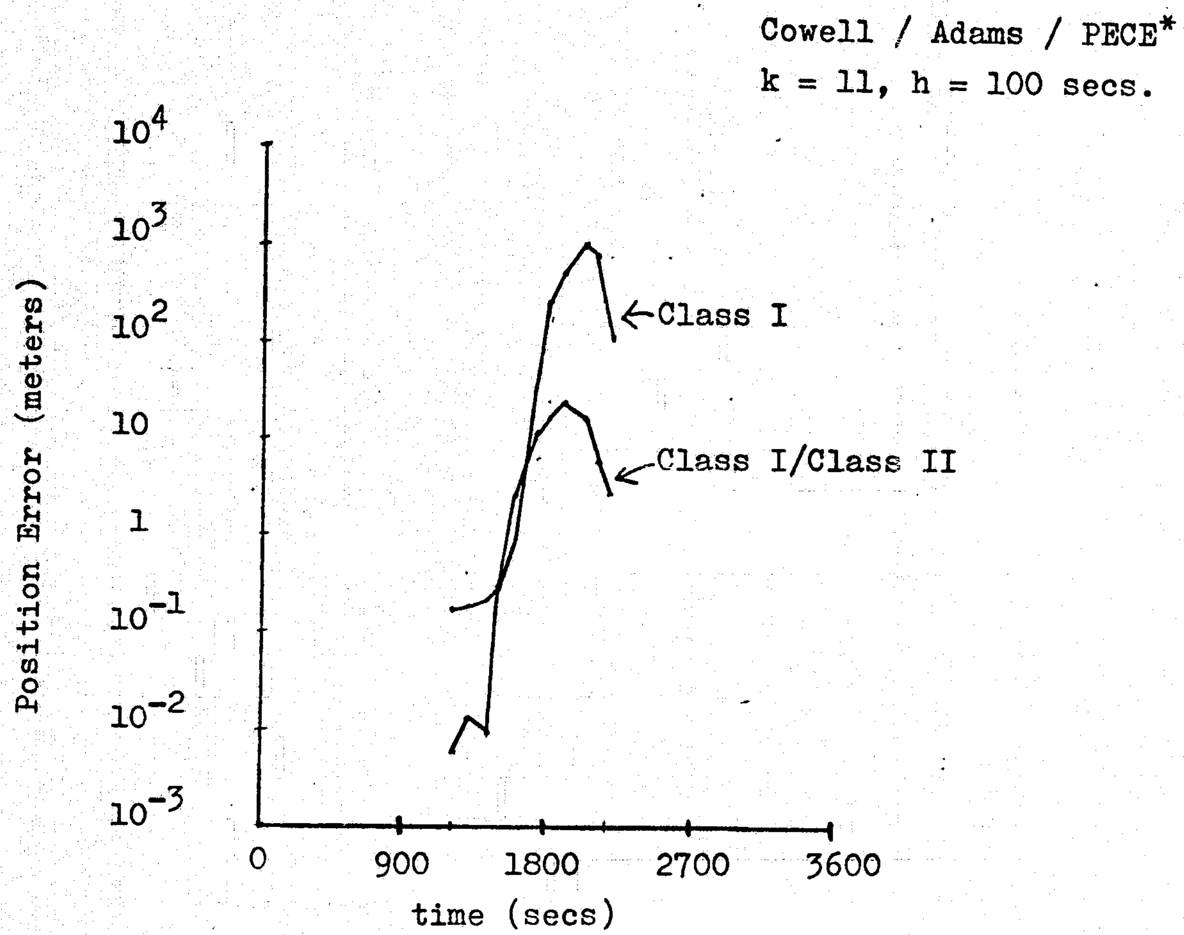


Figure 1-12 COMPARISON OF PROPAGATED ERROR IN CLASS I AND CLASS I/CLASS II INTEGRATION, TRAJECTORY 3.

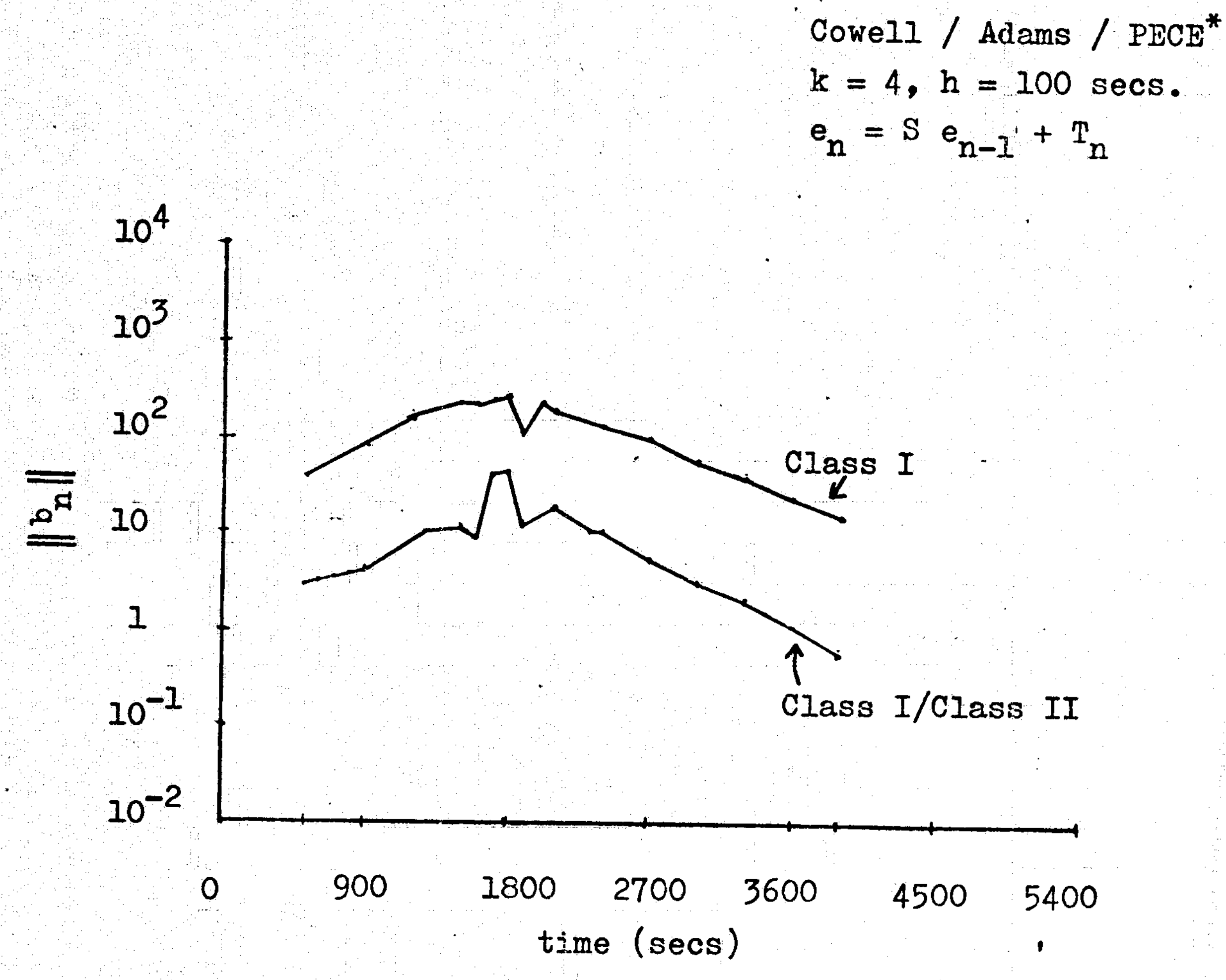


Figure 1-13 COMPARISON OF LOCAL ERROR $\|b_n\|$ IN CLASS I AND CLASS I/CLASS II INTEGRATION, TRAJECTORY 3.

Cowell / Adams / PECE*

$k = 11,\ h = 100$ secs.

$e_n = S\ e_{n-1} + b_n$

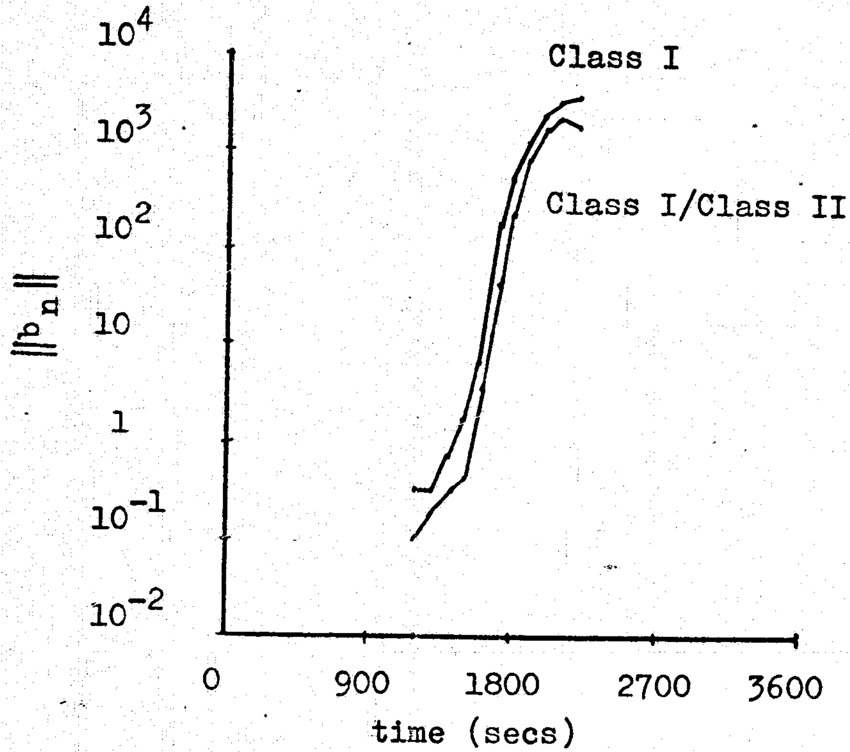


Figure 1-14 COMPARISON OF LOCAL ERROR $\|b_n\|$ IN CLASS I AND CLASS I/CLASS II INTEGRATION, TRAJECTORY 3.

1.4 Asymptotic Error in the Orbit Problem

Given the error equations

$$e_n = S_n e_{n-1} + b_n$$

where e_n, S_n and b_n vary according with the algorithm and are defined in the description of the general multistep Analyzer in this section.

Let the step-size, h, divide the period: $p = Nh$. After one revolution

$$e_N = \prod_{i=1}^{N} S_i e_0 + \sum_{i=1}^{N} \prod_{j=i}^{N} S_j b_i$$

where e_0 is the initial error vector. Let

$$\sum_{i=1}^{N} \prod_{j=i}^{N} S_j b_i = b$$

and let

$$\prod_{i=1}^{N} S_i = E .$$

Then, after m revolutions

$$e_{mN} = E^m e_0 + (E^{m-1} + E^{m-2} + \cdots + I)b \quad .$$

To analyze the propagated error, e_{mN}, for large m, it is convenient to begin with the analysis of $E^m e_0$ and to analyze E^m in cannonical form. This part of the analysis is taken, essentially, from [7].

Assume for the moment that E has all eigenvalues less than one in magnitude. For some non-singular matrix A, $A^{-1} EA = J$ is in block diagonal form, each of whose blocks, J_i, is in turn of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

where λ_i is an eigenvalue of E. Write $J_i = \lambda_i I + C$, where C is of order N_i and has ones on the superdiagonal and zeros elsewhere. It is clear that the C^r are as ones on the r^{th} super-diagonal and zeros elsewhere. Thus

$$J_i^k = \left(\lambda_i I + C\right)^k = \lambda_i^k I + \binom{k}{1} \lambda_i^{k-1} C + \cdots + \binom{k}{N_i - 1} \lambda_i^{k-N_i+1} C^{N_i-1} \tag{1-24}$$

The last term in (1-24) dominates. Thus

$$J_i^k X_i \sim \binom{k}{N_i - 1} \lambda_i^{k-N_i+1} C_i^{N_i-1} X_i \quad . \tag{1-25}$$

The definition of asymptotic equality requires the above assumption on the eigenvalues of E. Where the symbol means asymptotic equality [7]. $C_i^{N_i-1}$ is a matrix with a one in the upper right-hand corner and zeros elsewhere. $C^{N_i-1} X_i$ is non-zero if and only if X_i has a non-vanishing last component. It is then easy to show [7] that $C^{N_i-1} X_i$ is an eigenvector of J_i and the only eigenvector up scalar multiples.

It is clear that only the Jordan blocks with λ_i equal to the spectral radius and with order N_i, maximal, need be considered in the asymptotic behavior of $A^{-1} H^k A X$.

Since $EA = AJ$, there is exactly one column S_i of S, corresponding to each block J_i, which is an eigenvector of E. Any vector X can be decomposed uniquely in the directions of the columns of S. Let W_i denote the one in the direction of S_i. W_i is an eigenvector of E belonging to λ_i. It follows that

$$E^k X \sim \binom{k}{N_i - 1} \sum \lambda_i^{i-N_i+1} W_i \quad . \tag{1-26}$$

The sum is taken over all blocks with $|\lambda_i| = \bar{\lambda}$ and $N_i = \bar{N}$, where $\bar{\lambda}$ and $\bar{N}$ are maximal values of $|\lambda_i|$ and N_i.

This relationship is true for almost every X -- i.e., almost every e_0. (Because of round-off error, the statement is probably true for every e_0).

The assumption on the eigenvalues of E can be removed. Let E be arbitrary and divide E by a real scalar s. The following can easily be obtained

$$\left(\frac{1}{s} E \right)^k e_0 \sim \binom{k}{\bar{N}-1} \sum \frac{\lambda_i^{k-\bar{N}+1}}{s^k} W_i, \qquad k \to \infty \quad . \tag{1-27}$$

Relationship (1-27) is a statement of error growth in terms of eigenvectors of E.

Application to the Growth of the solution to the Homogeneous Error Equation $e_{mN}^{(1)} = E^m e_0$

In the periodic orbit problem, E (Cowell) has been found to have one real principal eigenvalue, λ_1, with magnitude greater than one, one eigenvalue $\lambda_2 = 1/\lambda_1$, and four principal eigenvalues equal to one. The principal eigenvalue in the case Cowell/Störmer/PE, $k = 11$, $h = 100$ secs., 2-hr. circular orbit is $\lambda_1 = 1.00008$. The large principal root points out the L-instability of the equation of motion. Matrix E is a product of large matrices and the multiplication introduces con-

siderable round-off error in the product. Examples of these eigenvalues are given below. In numerically stable cases, all non-principal eigenvalues are less than one in magnitude.

Applying the above analysis, we have that $\bar{N} = 1$ and that there is only one term in the sum in (1-27) with $\lambda_1 = \bar{\lambda}$, the large principal root

$$\left(\frac{E}{s} \right)^k \sim \frac{\lambda^k}{s^k} W_1 .$$

It is clear that the asymptotic error growth is exponential. However, it is important to understand that the large principal root is very close to 1. The situation closely approximates one in which there are six principal roots equal to one. Thus, it is important, for subasymptotic error growth to know the size of the Jordan blocks associated with all rpincipal eigenvalues. Assume that these Jordan blocks are of order one and two. The contribution to the error growth of a pair of eigenvalues equal to one is roughly

$$E^k e_0 \sim k W_i .$$

With this assumption, in all practical computations, the exponential character of the error growth is probably not apparent due to this consideration.

Next, it is important to consider the operator $T_k = E^k + E^{k-1} + \cdots + I$. Only a sketch of the possible analysis of this operator will be given. Each E^j can be put into canonical form and, as before, consider a Jordan block J_i^j, $k^j > N_i - 1$

$$J_i^k = \lambda_i^k I + \binom{k}{1} \lambda_i^{k-1} C + \cdots + \binom{k}{N_i - 1} \lambda^{k-N_i+1} C^{N_i-1}$$

$$J_i^{k-1} = \lambda_i^{k-1} I + \binom{k-1}{1} \lambda^{k-1} C + \cdots + \binom{k-1}{N_i - 1} \lambda^{k-N_i} C^{N_i-1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$J_i^{N_i-1} = \lambda_i^{N_i-1} I + \binom{N_i - 1}{1} \lambda^{k-N_i} C + \cdots + \binom{N_i - 1}{N_i - 1} C^{N_i-1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Assume λ_i real, $\lambda_i > 1$ and consider the operator

$$\frac{T_k}{\lambda^{k-N_i+1} \binom{k}{N_i - 1}} .$$

After dividing the above expansions by

$$\lambda^{k-N_i+1} \binom{k}{N_i - 1} = a_k ,$$

the last terms of each of the expansions given are dominant.

$$\frac{\binom{k}{N_i - 1} \lambda^{k-N+1}}{a_k} = 1$$

$$\frac{\binom{k-1}{N_i - 1} \lambda^{k-N_i}}{a_k} \longrightarrow \frac{1}{\lambda}$$

$$\frac{\binom{k-2}{N_i - 1} \lambda^{k-N_i-1}}{a_k} \longrightarrow \frac{1}{\lambda^2}$$

. . . .

The series given by the total sum of the J_i^j $(j \geq N_i - 1)$ should be investigated and the series

$$1 + \frac{\binom{k-1}{N_i - 1} \lambda^{k-N_i}}{a_k} + \frac{\binom{k-2}{N_i - 1} \lambda^{k-N_i-1}}{a_k} + \cdots$$

given by the sum of dominant terms. If the total series is representable by the sum of the limits given, we have, summing these limits,

$$1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^{k-N_i-1}} .$$

The series is bounded by $1/(1 - 1/\lambda)$. If this is

the case there is apparently no essential difference in the order of the asymptotic rate of error growth in the two solutions, $E^k e_0$ and $(E^{k-1} + E^{k-2} + \ldots + I)b$. Further analysis should proceed from this point. It is important to determine the order (N_i) of the Jordan blocks associated with the six principal eigenvalues λ_i, as well as to understand the convergence of the series given by the sum of the expansions of the J_i^j. A suitable computer program for analysis of the eigenvectors of an arbitrary real matrix is obtainable from Argonne National Laboratory.

The spectra of some error operators

It had been conjectured that the eigenvalues of the error operator S_n, for small step-size and applied to a circular trajectory will closely approximate the roots of the characteristic polynomial, as generated by program D1/D1.5 if the eigenvalues $2\mu/r^3$ and $-\mu/r^3$ are substituted into that polynomial. The reason behind this is that eigenvalues of J_n are constant on the circle and the J_{n-i} appearing in the error equations can be all approximately diagonalized by the same transformation.

The conjecture was tested on a 2-hr. circular trajectory. The subroutine used to compute the eigenvalues of S_n was obtained from Atomics International, North American Aviation Corp. The 11^{th} order Adams predictor and corrector were used. The resulting dimension of S_n

was $11 \times 2 \times 3 = 66$. The first and last factors are associated with the step-number of the integrator and the number of equations, respectively, the factor of two results from the fact that both e_n and e_n^P are involved in the propagation.

The eigenvalues of S_n are independent of n in the circular problem. It was shown in the Final Report for the last contract period [6], that the characteristic polynomial of the corrector-only algorithm is invariant under a similarity transformation of the constant J_{n-i}. This fact undoubtedly holds for any algorithm and, moreover, it appears true that the characteristic polynomial of S_n (for any particular algorithm structure is identical with the characteristic polynomials as derived in the abovementioned report. Program D1/D1.5 [6] was used in checking out the program which constructs the matrices S_n.

A comparison of the eigenvalues of S_n, Cowell, Class II algorithm PEC, $k = 11$, Adams integration, for the 2-hour circular problem $h = 100$ secs. is given below with the roots of the characteristic polynomials as generated by Program D1/D1.5 by substitution

$$\bar{h} = \frac{2h^2 \mu}{r^3} \qquad \text{and} \qquad \bar{h} = - \frac{h^2 \mu}{r^3} .$$

	Non-zero Eigenvalues $a + ib$ of S_n (local operator)		Non-zero Roots Given by Program D1/D1.5	
			$\bar{h} = 2\mu h^2/r^3$	
	a	b	a	b
Principal Roots	(1.131	0	1.133	0
	(.884	0	.833	0
	-.858	.563	-.864	.571
	-.858	.563	-.864	-.571
	-.116	.679	-.113	.684
	-.116	-.678	-.113	.684
	.179	.465	.182	.465
	.179	-.465	.182	-.465
	.335	.132	.336	.133
	.335	-.132	.336	-.133
	.291	.282	.292	.281
	.292	-.281	.292	-.281
	.348	0	.347	0
			$\bar{h} = -\mu h^2/r^3$	
Principal Roots	(.996	.087	.996	.088
	(.996	.087	.996	-.088
	-.993	0	-.997	0
	.416	.674	-.416	.658
	.416	-.647	-.416	-.658
	.049	.543	.050	.544
	.049	-.543	.050	-.544
	.303	.202	.303	.198
	.303	-.202	.303	-.198
	.329	.064	.330	.064
	.329	-.064	.330	-.064
	.229	.351	.230	.355
	.229	-.351	.230	-.355
	27 roots of zero			

This last group of non-zero eigenvalues of S_n is repeated, due to the fact that $-\mu/r^3$ is a double root of J_{n-i}. Thus, the form of the characteristic polynomial is that derived in [6]. The slight differences in roots is computed from S_n and from Program D1/D1.5, arise primarily because J_{n-i} is non-constant in S_n and a single transfor-

mation T approximately, but not completely, diagonalizes $TS_n T^{-1}$. If the eccentricity, $e \neq 0$, then the differences are more marked, because not only are the J_{n-i} not all diagonalized by the same transformation, but also, the eigenvalues of J_{n-i} are a function of time. S_n has twenty-seven roots of zero.

Composite operators

Examples are given below of the sets of principal roots of the composite error operator $E = \Pi S_i$, the product taken over one complete revolution of a Keplerian orbit. The spectrum of E for the above problem with algorithm PE is (to five decimal places):

a	b
.99992	0
1.00008	0
1.00000	0
1.00000	0
1.00000	0
1.00000	0

All other eigenvalues are less than 11^{-8} in magnitude. E is a 33×33 matrix.

The spectrum of E for the problem: Cowell, Class I, Adams integrator, algorithm PECE* = PECE, $k = 4$, $h = 100$, 2-hr. Keplerian orbit, $e = .01$ (to five place accuracy) is

a	b
.98180	0
1.01830	0
1.00030	.00020
1.00030	-.00020
1.00000	0
1.00000	0

All other eigenvalues are less than 10^{-12} in magnitude, E is a 48×48 matrix.

The spectrum of E for the problem: Cowell, Class II, Adams integrators, algorithm PECE* = PECE, $k = 4$, $h = 100$, 2-hr. Keplerian orbit, $e = .01$ (to five place accuracy) is

a	b
.99460	0
1.00540	0
1.00000	0
1.00000	0
1.00000	0
1.00000	0

All other eigenvalues are less than 10^{-13} in magnitude, E is a 24×24 matrix.

In all cases, the first two eigenvalues λ_1 and λ_2 satisfy $\lambda_2 = \lambda_1^{-1}$.

2. Reduction of Propagated Error by Placement of Extraneous Roots

2.1 Method Definition and Coefficients

Four types of corrector methods have been tested against the Adams' methods. These methods are: the W(r) methods, the RW(r) methods, the WA(r) methods and the WB(r) methods. All methods are of Class II.

The coefficients of these methods have been generated in Program A-B and used in both the summed and unsummed two-body integrator.

Given a corrector for $\ddot{y} = f(t,y)$:

$$\sum_{i=0}^{k} a_i \, y_{n-i} + h^2 \sum_{i=0}^{k} b_i \, y_{n-i} = 0, \qquad a_0 = -1 \quad , \qquad (2\text{-}1)$$

form the polynomial:

$$P(x) = \sum_{i=0}^{k} a_i \, x^{k-i} \quad .$$

$P(x) = 0$ has a double root of 1. The other roots are called extraneous roots, arising from the fact that the second-order equation is replaced by the k^{th} order degree difference equation.

Method (2-1) is strongly stable if the extraneous roots of $P(x) = 0$ are within the unit circle.

Placement of the extraneous roots of $P(x)$ was guided by results contained in Appendix B, resulting in the four types of methods mentioned above. These extraneous roots determine the a_i. The b_i are determined to yield the desired order $(k + 1)$.

<u>The</u> $W(r)$ <u>methods</u>

In the $W(r)$ methods, the extraneous roots are restricted in an r, θ arc, $90^\circ \leq \theta \leq 270^\circ$. The $W(r)$ methods tested are of orders 7, 8, 9, 10, 11, 12, 13 and $r = .6, .7, .8, .2$ and $.3$.

<u>The</u> $RW(r)$ <u>methods</u>

In the $RW(r)$ methods, the roots are restricted in the region of the unit circle with $140^\circ \leq \theta \leq 220^\circ$. The $RW(r)$ methods tested are of orders 7, 8, 9, 10, 11, 12, 13 and $r = .2, .3$ and $.4$.

<u>The</u> $WA(r)$ <u>methods</u>

The roots of the $WA(r)$ methods are the same as those of the $W(r)$ methods, of being two orders lower. Two additional zero roots are added. The $WA(r)$ methods tested are of order 13 with $r = .3, .5$ and of

order 12 with $r = .3$.

The WB(r) method

The roots of the WB(r) methods are the same as those of the W(r) methods, of four orders lower, with four additional zero roots. The WB(r) methods tested are of order 13 with $r = .3$ and $.5$.

For each method tested, all roots of $P(x)$ are given, as well as coefficients a_i, b_i (of 2-1) in the following computer output. Roots are specified by modulus and argument under the heading "PRELIMINARY ROOT." Conjugates must be added to the set of input complex roots. Coefficients appear in the usual order $a_0, a_1, \ldots, a_k$ and $b_0, b_1, \ldots, b_k$ as given in the description of the coefficient generator used, Program A-B [6]. Roots of the $h = 0$ characteristic polynomial $P(x)$ are recomputed and output in the form $x + iy$.

METHOD W(.7)

ORDER 7

```
INPUT

STEP NUMBER  K =      6
METHOD CLASS  J =     2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -4
KR=-INPUT NP =    4.  NP IS RESET TO K-2 =    4
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    7.0000000000000000D-01   9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    7.0000000000000000D-01   1.3500000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                   BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                   BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


COEFFICIENTS ARE

  -1.0000000000000000D+00   1.0100506413385675D+00  -1.0121782272163787D-04   4.8497532588563887D-01
  -2.4394956365738280D-01  -4.8751857441019286D-03  -2.4010000000000000D-01

   6.7602123354553634D-02   9.8638387870917903D-01   8.5775576372659355D-01   1.0774699883703773D+00
   4.1588299823126246D-01   2.7731646141744597D-01   1.2713265741709178D-02
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                          9.9999999999998344D-01   0.
   3.2756250448903002D-08  -7.0000000000000389D-01  -4.9497471205185559D-01  -4.9497478155000217D-01
  -4.9497471208714391D-01   4.9497478157402103D-01   3.2756427633461913D-08   6.9999999999999923D-01
   9-TH TRUNCATION-COEFFICIENT =    2.12847058D-03
  10-TH TRUNCATION-COEFFICIENT =    7.61394518D-03
  11-TH TRUNCATION-COEFFICIENT =    1.39223280D-02

THETA =   -0.                      ,  B-SUB-THETA =    -0.
```

2-4

STABLE

METHOD RW(.3)

ORDER 7

```
INPUT

STEP NUMBER  K =       6
METHOD CLASS  J =       2
NUMBER OF PREDETERMINE PARAMETERS  NP =      -4
KR=-INPUT NP =     4.  NP IS RESET TO K-2 =     4
    1-TH PRELIMINARY ROOT IS     1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS     1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS     3.0000000000000000D-01   1.4000000000000000D+02
    4-TH PRELIMINARY ROOT IS     3.0000000000000000D-01   1.6000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                          TO   -0.                   BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                          TO   -0.                   BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   9.7655780680275614D-01   6.0773973832817496D-01  -2.3726269445237030D-01
  -2.6302505329080886D-01  -7.5909797387751947D-02  -8.1000000000000000D-03

   6.7848436361191320D-02   9.8870241663946886D-01   8.4397507176856915D-01   5.8307256742328276D-01
   4.5938333506690307D-02   3.5922510391853002D-02  -2.6631047146746515D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                       9.9999999999999980D-01   0.
  -2.8190777699879962D-01  -1.0260606644987038D-01  -2.8190777699879967D-01   1.0260606644987042D-01
  -2.2981331889874347D-01   1.9283629963454930D-01  -2.2981331889874225D-01  -1.9283629963454753D-01
    9-TH TRUNCATION-COEFFICIENT =    2.63871851D-03
   10-TH TRUNCATION-COEFFICIENT =    8.90113374D-03
   11-TH TRUNCATION-COEFFICIENT =    1.56523826D-02

THETA =    -0.                       ,  B-SUB-THETA =    -0.
```

2-5

STABLE

METHOD W(.7)

ORDER 8

```
INPUT

STEP NUMBER  K =      7
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =    -5
KR=-INPUT NP =    5.  NP IS RESET TO K-2 =    5
     1-TH PRELIMINARY ROOT IS     1.0000000000000000D+00  -0.
     2-TH PRELIMINARY ROOT IS     1.0000000000000000D+00  -0.
     3-TH PRELIMINARY ROOT IS    -7.0000000000000000D-01  -0.
     4-TH PRELIMINARY ROOT IS     7.0000000000000000D-01   9.0000000000000000D+01
     5-TH PRELIMINARY ROOT IS     7.0000000000000000D-01   1.3500000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                          TO    -0.                    BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                          TO    -0.                    BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   3.1005664133856749D-01   7.0693423111427561D-01   4.8490447340973372D-01
   8.9533164462564413D-02  -1.7983988030426989D-01  -2.4351263002087135D-01  -1.6807000000000000D-01

   6.3983723369645289D-02   1.0590341649517250D+00   1.4722380791399436D+00   1.8045430224507848D+00
   1.0434679906187345D+00   6.4442095986240492D-01   1.8150598883956294D-01   1.2517686004104769D-02
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                         9.9999999999898467D-01   0.
  -6.9999999998713696D-01   0.                         3.2756250448900154D-08  -7.0000000000003809D-01
  -4.9497471205185559D-01  -4.9497478155000217D-01  -4.9497471208714391D-01   4.9497478157402103D-01
   3.2756427633461913D-08   6.9999999999999923D-01
   10-TH TRUNCATION-COEFFICIENT =    2.40777744D-03
   11-TH TRUNCATION-COEFFICIENT =    9.12807156D-03
   12-TH TRUNCATION-COEFFICIENT =    1.80779943D-02

THETA =    -0.                     B-SUB-THETA =    -0.
```

2-6

STABLE

METHOD RW(.3)

ORDER 8

```
INPUT

STEP NUMBER  K =      7
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -5
KR=-INPUT NP =    5.  NP IS RESET TO K-2 =     5
     1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
     2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
     3-TH PRELIMINARY ROOT IS   -3.0000000000000000D-01  -0.
     4-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.4000000000000000D+02
     5-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.6000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                    TO   -0.                    BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                    TO   -0.                    BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


COEFFICIENTS ARE

  -1.0000000000000000D+00   6.7655780680275614D-01   9.0070708036900181D-01  -5.4940772953917813D-02
  -3.3420386162651995D-01  -1.5481731337499460D-01  -3.0872939216325584D-02  -2.4300000000000000D-03

   6.4418509567850655D-02   1.0330694082081839D+00   1.0685487813267718D+00   9.5632678134325019D-01
   1.0079841134427845D-01   1.2174102587749811D-01  -1.5898690074998088D-02   2.6314026538660813D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                        9.9999999999999970D-01   0.
  -2.9999999999999997D-01   0.                       -2.8190777769987962D-01  -1.0266066449870380D-01
  -2.8190777769987967D-01   1.0266066449870420D-01  -2.2981331889874347D-01   1.9283629963454930D-01
  -2.2981331889874225D-01  -1.9283629963454753D-01
   10-TH TRUNCATION-COEFFICIENT =    2.20402313D-03
   11-TH TRUNCATION-COEFFICIENT =    8.55728016D-03
   12-TH TRUNCATION-COEFFICIENT =    1.72486507D-02

THETA =   -0.                   ,  B-SUB-THETA =    -0.
RUN COMPLETED
```


STABLE

METHOD W(.6)

ORDER 9

```
INPUT

STEP NUMBER  K =      8
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -5
KR=-INPUT NP =     5.  NP IS RESET TO K-2 =     6
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    6.0000000000000000D-01   9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    6.0000000000000000D-01   1.2000000000000000D+02
    5-TH PRELIMINARY ROOT IS    6.0000000000000000D-01   1.5000000000000000D+02
NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                           TO   -0.                        BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                           TO   -0.                        BY   -0.
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

 -1.0000000000000000D+00   3.6076968324836130D-01
  4.3679765723054809D-02  -1.6614260778397707D-01   5.7492253028920976D-01   5.8760009662940318D-01
 -4.6656000000000000D-02                           -2.3504121905503960D-01  -1.1913224905101238D-01

  6.1615990004952178D-02   1.0745492510070443D+00
  8.7071830047371983D-01   3.2956263519280766D-01   1.3613362978847526D+00   1.9721199324477952D+00
  5.1723267838643527D-04                            1.4550146470132238D-01   7.9467074830195934D-02
METHOD IS STABLE

ROOTS ARE

  1.0000000000000000D+00   0.
 -2.9999996757953603D-01  -5.1961526096858243D-01   9.9999999999819220D-01   0.
 -2.9999996757954388D-01   5.1961526098862014D-01  -5.1961521886998185D-01  -3.0000004052499615D-01
  2.8076933502558996D-08   5.9999999999999876D-01  -5.1961521886562319D-01   3.0000004052512960D-01
  11-TH TRUNCATION-COEFFICIENT =    1.82820156D-03    2.8076937971538783D-08  -5.9999999999999934D-01
  12-TH TRUNCATION-COEFFICIENT =    8.08276156D-03
  13-TH TRUNCATION-COEFFICIENT =    1.84536548D-02
THETA =   -0.
                                    ,  B-SUB-THETA =   -0.
```

STABLE

METHOD W(.7)

ORDER 9

```
INPUT

STEP NUMBER  K =      8
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =    -5
KR=-INPUT NP =     5.  NP IS RESET TO K-2 =     6
    1-TH PRELIMINARY ROOT IS    1.000000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.000000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    7.000000000000000000D-01   9.000000000000000000D+01
    4-TH PRELIMINARY ROOT IS    7.000000000000000000D-01   1.200000000000000000D+02
    5-TH PRELIMINARY ROOT IS    7.000000000000000000D-01   1.500000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                        TO   -0.                        BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                        TO   -0.                        BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.000000000000000000D+00   8.756463045642151 5D-02   5.061660986013426 8D-01   8.507873048764156 7D-01
   2.935032987794055 6D-01  -6.103179110294961 8D-02  -3.354628093832226 2D-01  -2.238777322274131 9D-01
  -1.176490000000000 0D-01

   6.095992038796738 0D-02   1.096951269581099 0D+00   1.644731465515585 3D+00   2.563089330024688 6D+00
   1.536865423506916 6D+00   1.354553006558356 1D+00   3.916565836536485 7D-01   1.647705570676262 8D-01
   4.739066150599501 2D-03
METHOD IS STABLE

ROOTS ARE

   1.000000000000000000D+00   0.                          9.999999999797414 0D-01   0.
  -3.499999621749283 0D-01  -6.062178044870489 6D-01  -6.062177553483121 6D-01  -3.500000472791621 7D-01
  -3.499999621761345 3D-01   6.062178044867235 0D-01  -6.062177553432270 6D-01   3.500000472793178 7D-01
   3.275642825299549 6D-08   6.999999999999985 6D-01   3.275642763346191 4D-08  -6.999999999999992 3D-01
   11-TH TRUNCATION-COEFFICIENT =    1.65898617D-03
   12-TH TRUNCATION-COEFFICIENT =    7.43128899D-03
   13-TH TRUNCATION-COEFFICIENT =    1.71337028D-02

THETA =    -0.                          ,  B-SUB-THETA =    -0.
```

STABLE

```
INPUT                                                                                  METHOD W(.8)

                                                                                       ORDER 9
STEP NUMBER  K =       8
METHOD CLASS  J =       2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -5
KR=-INPUT NP =      5.   NP IS RESET TO K-2 =     6
    1-TH PRELIMINARY ROOT IS     1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS     1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS     8.0000000000000000D-01   9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS     8.0000000000000000D-01   1.2000000000000000D+02
    5-TH PRELIMINARY ROOT IS     8.0000000000000000D-01   1.5000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                      BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                      BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -1.8564042233551827D-01   3.4276866117936073D-01   1.0737642670547231D+00
   6.2847933425987224D-01   1.6363762028760647D-01  -4.0991516345741595D-01  -3.7095031698862828D-01
  -2.6214400000000000D-01

   6.0167189499177612D-02   1.1202407546552123D+00   1.9310661080835967D+00   3.2656779882308926D+00
   2.4049768344365756D+00   2.1669752803441649D+00   8.1052251800321722D-01   3.3439850510402210D-01
   1.3357199486746098D-02
METHOD IS STABLE                                                                                  STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                         9.9999999985059797D-01   0.
  -3.9999995677553263D-01  -6.9282034798551385D-01  -6.9282029182664247D-01  -4.0000005403332819D-01
  -3.9999995677272517D-01   6.9282034798482665D-01  -6.9282029182083092D-01   4.0000005403350613D-01
   3.7435918003412001D-08   7.9999999999999835D-01   3.7435917295385044D-08  -7.9999999999999912D-01
   11-TH TRUNCATION-COEFFICIENT =     1.36883988D-03
   12-TH TRUNCATION-COEFFICIENT =     6.34702302D-03
   13-TH TRUNCATION-COEFFICIENT =     1.49996322D-02

THETA =    -0.                          ,  B-SUB-THETA =    -0.
```

```
INPUT                                                                     METHOD RW(.2)

STEP NUMBER  K =      8                                                   ORDER 9
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =      -5
KR=-INPUT NP =    -5.  NP IS RESET TO K-2 =     6
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    3-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.4000000000000000D+02
    4-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.5400000000000000D+02
    5-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.6800000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                       TO    -0.                BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                       TO    -0.                BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   9.4280560396000448D-01   6.2367298737126465D-01  -2.0344044225237449D-01
  -2.5498912563741193D-01  -9.0111902286797641D-02  -1.6309610121021070D-02  -1.5635110336639928D-03
  -6.4000000000000000D-05

   6.2833500771928661D-02   1.0283440409683009D+00   7.5745838188625439D-01   8.5137203505034689D-01
  -1.7127378938518245D-01   2.3229632656638036D-01  -8.2910208435374534D-02   2.1148776124848361D-02
  -2.2920639469359009D-03
METHOD IS STABLE                                                                               STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                        1.0000000000000000D+00   0.
  -1.5320887926582816D-01  -1.2855753308969835D-01  -1.7975880223964759D-01  -8.7674243751330984D-02
  -1.9562951651480368D-01  -4.1582355251980341D-02  -1.9562951651445311D-01   4.1582355251840406D-02
  -1.7975880216315297D-01   8.7674243750419151D-02  -1.5320887926582817D-01   1.2855753308969835D-01
   11-TH TRUNCATION-COEFFICIENT =    2.01979786D-03
   12-TH TRUNCATION-COEFFICIENT =    8.86339386D-03
   13-TH TRUNCATION-COEFFICIENT =    2.01175859D-02

THETA =    -0.                   ,  B-SUB-THETA =    -0.
```

METHOD RW(.3)

ORDER 9

```
INPUT

STEP NUMBER  K =       8
METHOD CLASS  J =       2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -5
KR=-INPUT NP =    5.  NP IS RESET TO K-2 =     6
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.4000000000000000D+02
    4-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.5400000000000000D+02
    5-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.6800000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                        TO   -0.                  BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                        TO   -0.                  BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

 -1.0000000000000000D+00   4.1420840594000671D-01   1.0674726275253522D+00   1.9151743937859331D-01
 -3.4165633554678711D-01  -2.4501709875553395D-01  -7.4409126629745204D-02  -1.1386911911885946D-02
 -7.2900000000000000D-04

  6.1639818643488496D-02   1.0721709319147159D+00   1.2626945919284449D+00   1.4023124905670520D+00
  1.6750400152213113D-01   3.0925400675322174D-01  -6.0614329796025200D-02   2.0884948207195312D-02
 -2.0883551695114084D-03
METHOD IS STABLE

ROOTS ARE

  1.0000000000000000D+00   0.                       9.9999999999999621D-01   0.
 -2.6963820335947381D-01  -1.3151136562698875D-01  -2.9344427477178474D-01  -6.2373532877824302D-02
 -2.9344427477178528D-01   6.2373532877824910D-02  -2.6963820335936174D-01   1.3151136562417963D-01
 -2.2981331889874225D-01   1.9283629963454753D-01  -2.2981331889874225D-01  -1.9283629963454753D-01
  11-TH TRUNCATION-COEFFICIENT =    1.87792704D-03
  12-TH TRUNCATION-COEFFICIENT =    8.25039814D-03
  13-TH TRUNCATION-COEFFICIENT =    1.87492015D-02

THETA =   -0.                     ,  B-SUB-THETA =   -0.
```

STABLE

METHOD RW(.4)

ORDER 9

```
INPUT

STEP NUMBER  K =        8
METHOD CLASS  J =        2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -5
KR=-INPUT NP =     5.  NP IS RESET TO K-2 =      6
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.4000000000000000D+02
    4-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.5400000000000000D+02
    5-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.6800000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                     TO   -0.                        BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                     TO   -0.                        BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

 -1.0000000000000000D+00  -1.1438879207999105D-01   1.2659143653250765D+00   7.8991640055116605D-01
 -2.3407884377118232D-01  -4.4743336808873207D-01  -2.0989740885908935D-01  -4.5936353077247771D-02
 -4.0960000000000000D-03

  6.0524458003490042D-02   1.1147407459489668D+00   1.7910676388324784D+00   2.1815160393364370D+00
  7.7332304759693530D-01   5.2854210446944865D-01  -2.4395156316468899D-03   2.5454490123804239D-02
 -1.7732846355301255D-03
METHOD IS STABLE                                                                                    STABLE

ROOTS ARE

  1.0000000000000000D+00   0.                         9.9999999999984677D-01   0.
 -3.5951760452072242D-01  -1.7534848752219282D-01  -3.9125903302904632D-01  -8.3164710503765736D-02
 -3.9125903302904704D-01   8.3164710503766546D-02  -3.5951760447914898D-01   1.7534848749890618D-01
 -3.0641775853165634D-01   2.5711506617939671D-01  -3.0641775853165634D-01  -2.5711506617939671D-01
  11-TH TRUNCATION-COEFFICIENT =    1.74912982D-03
  12-TH TRUNCATION-COEFFICIENT =    7.69462662D-03
  13-TH TRUNCATION-COEFFICIENT =    1.75092021D-02

THETA =    -0.                     ,  B-SUB-THETA =    -0.
```

METHOD W(.6)

ORDER 10


```
INPUT

STEP NUMBER  K =       9
METHOD CLASS  J =       2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -6
KR=-INPUT NP =     6.  NP IS RESET TO K-2 =     7
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    3-TH PRELIMINARY ROOT IS   -6.0000000000000000D-01   -0.
    4-TH PRELIMINARY ROOT IS    6.0000000000000000D-01    9.0000000000000000D+01
    5-TH PRELIMINARY ROOT IS    6.0000000000000000D-01    1.2000000000000000D+02
    6-TH PRELIMINARY ROOT IS    6.0000000000000000D-01    1.5000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                    BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                    BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   -2.3923031675163870D-01    7.9138434023822654D-01    9.3255361480292904D-01
   3.9623982370069672D-01   -1.3993474835014419D-01   -3.3472678372542585D-01   -2.6015698048403614D-01
  -1.1813534943060743D-01   -2.7993600000000000D-02

   5.8690867501863522D-02    1.1378449475368535D+00    1.9007614383813875D+00    3.0346320014296938D+00
   1.6854248245658263D+00    1.7205590506536102D+00    3.9752875556595999D-01    2.7207236375858096D-01
   2.1871375049606095D-02    3.2354621100205168D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00    0.                         9.9999999996483352D-01    0.
  -6.0000000000000000D-01    0.                        -2.9999996757953603D-01   -5.1961526098658243D-01
  -5.1961521886998135D-01   -3.0000004052499615D-01    -2.9999996757954388D-01    5.1961526098862014D-01
  -5.1961521886562319D-01    3.0000004052512960D-01     2.8076938502558980D-08    5.9999999999999876D-01
   2.8076937971538767D-08   -5.9999999999999934D-01
   12-TH TRUNCATION-COEFFICIENT =    1.59756880D-03
   13-TH TRUNCATION-COEFFICIENT =    7.80892369D-03
   14-TH TRUNCATION-COEFFICIENT =    1.96996273D-02

THETA =    -0.                      ,  B-SUB-THETA =    -0.
```

STABLE

METHOD W(.7)

ORDER 10

```
INPUT

STEP NUMBER  K =        9
METHOD CLASS  J =        2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -6
KR=-INPUT NP =      6.  NP IS RESET TO K-2 =     7
    1-TH PRELIMINARY ROOT IS     1.000000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS     1.000000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    -7.000000000000000000D-01  -0.
    4-TH PRELIMINARY ROOT IS     7.000000000000000000D-01   9.000000000000000000D+01
    5-TH PRELIMINARY ROOT IS     7.000000000000000000D-01   1.200000000000000000D+02
    6-TH PRELIMINARY ROOT IS     7.000000000000000000D-01   1.500000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                         TO    -0.                    BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                         TO    -0.                    BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.000000000000000000D+00  -6.12435369543578490D-01   5.67461339920837740D-01   1.20510357389735550+00
   6.89054412192896530D-01   1.44420518042634280D-01  -3.76185063155287350D-01  -4.58701698795669020D-01
  -2.74363412559189240D-01  -8.23543000000000000D-02

   5.81396438965191890D-02   1.16500570227570990D+00   2.31106740053021980D+00   3.95130458116724640D+00
   2.97567311660172850D+00   2.78571364093567120D+00   1.10294046316284980D+00   5.40460119457315160D-01
   9.46959676749041730D-02   6.13762279686784210D-03
METHOD IS STABLE

ROOTS ARE

   1.000000000000000000D+00   0.                          1.00000000000004740D+00   0.
  -7.000000000000000000D-01   0.                         -3.49999962174928300D-01  -6.06217804487048960D-01
  -6.06217755348312160D-01  -3.50000047279162170D-01  -3.49999962176134530D-01   6.06217804486723500D-01
  -6.06217755343227060D-01   3.50000047279317870D-01   3.27564282529854780D-08   6.99999999999998560D-01
   3.27564276334618970D-08  -6.99999999999999230D-01
   12-TH TRUNCATION-COEFFICIENT =    1.60093324D-03
   13-TH TRUNCATION-COEFFICIENT =    7.77517372D-03
   14-TH TRUNCATION-COEFFICIENT =    1.95185018D-02

THETA =    -0.                   ,  B-SUB-THETA =    -0.
```

STABLE

METHOD W(.8)

ORDER 10

```
INPUT

STEP NUMBER  K =      9
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -6
KR=-INPUT NP =     6.   NP IS RESET TO K-2 =    7
    1-TH PRELIMINARY ROOT IS     1.0000000000000000D+00   -0.
    2-TH PRELIMINARY ROOT IS     1.0000000000000000D+00   -0.
    3-TH PRELIMINARY ROOT IS    -8.0000000000000000D-01   -0.
    4-TH PRELIMINARY ROOT IS     8.0000000000000000D-01    9.0000000000000000D+01
    5-TH PRELIMINARY ROOT IS     8.0000000000000000D-01    1.2000000000000000D+02
    6-TH PRELIMINARY ROOT IS     8.0000000000000000D-01    1.5000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                          TO   -0.                      BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                          TO   -0.                      BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -9.8564042233551827D-01   1.9425632331094611D-01   1.3479792159982116D+00
   1.4874907639036507D+00   6.8642108769550426D-01  -2.6300506722733078D-01  -6.9888244775456105D-01
  -5.5890425359090263D-01  -2.0971520000000000D-01

   5.7723277723128128D-02   1.1905657122389997D+00   2.7385578878699851D+00   5.0174994638859267D+00
   4.7070663412390546D+00   4.4014096316756604D+00   2.3371341530903924D+00   1.0715173434443773D+00
   2.5870079758551843D-01   1.3149671365446363D-02
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                                 1.0000000000000047D+00   0.
  -7.9999999999985052D-01   0.                                -3.9999995677553283D-01  -6.9282034798551385D-01
  -6.9282029182664247D-01  -4.0000005403332819D-01  -3.9999995677272517D-01   6.9282034798482685D-01
  [illegible]82029182083092D-01   4.0000005403350613D-01   3.7435918003411977D-08   7.9999999999999835D-01
   3.7435917295385027D-08  -7.9999999999999912D-01
   12-TH TRUNCATION-COEFFICIENT =     1.70587832D-03
   13-TH TRUNCATION-COEFFICIENT =     8.15970719D-03
   14-TH TRUNCATION-COEFFICIENT =     2.02361972D-02

THETA =    -0.                      ,  B-SUB-THETA =    -0.
```

2-16

STABLE

METHOD RW(.2)

ORDER 10

```
INPUT

STEP NUMBER  K =       9
METHOD CLASS  J =       2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -6
KR=-INPUT NP =     6.  NP IS RESET TO K-2 =     7
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS   -2.0000000000000000D-01  -0.
    4-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.4000000000000000D+02
    5-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.5400000000000000D+02
    6-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.6800000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                           TO    -0.                        BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                           TO    -0.                        BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   7.4280560396000448D-01   8.1223410816326554D-01  -7.8705844778121559D-02
  -2.9567721408788683D-01  -1.4110972741428003D-01  -3.4331990578380599D-02  -4.8254330578682069D-03
  -3.7670220673279857D-04  -1.2800000000000000D-05

   6.0409743335989418D-02   1.0627245580461399D+00   8.7587192238610182D-01   1.2064593360464942D+00
  -3.0639781930345774D-01   5.0343400561768854D-01  -2.4004656774099491D-01   9.1822002131586217D-02
  -1.9876125664519419D-02   1.9653446465520632D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                        1.0000000000000058D+00   0.
  -1.9999999999999980D-01   0.                       -1.5320887926582816D-01  -1.2855753308969835D-01
  -1.7975880223964759D-01  -8.7674243751330984D-02  -1.9562951651480868D-01  -4.1582355251980341D-02
  -1.9562951651445311D-01   4.1582355251840406D-02  -1.7975880216315297D-01   8.7674243750419151D-02
  -1.5320887926582817D-01   1.2855753308969835D-01
   12-TH TRUNCATION-COEFFICIENT =   1.74896203D-03
   13-TH TRUNCATION-COEFFICIENT =   8.56494283D-03
   14-TH TRUNCATION-COEFFICIENT =   2.16299108D-02

THETA =    -0.                    ,  B-SUB-THETA =    -0.
```

2-17

STABL

```
INPUT

STEP NUMBER  K =      9
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -6
KR=-INPUT NP =     6.  NP IS RESET TO K-2 =     7
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS   -3.0000000000000000D-01  -0.
    4-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.4000000000000000D+02
    5-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.5400000000000000D+02
    6-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.6800000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                          TO   -0.                 BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                          TO   -0.                 BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   1.1420840594000671D-01   1.1917351493073542D+00   5.1175922763619897D-01
  -2.8420110373320911D-01  -3.4751399941957008D-01  -1.4791425625640539D-01  -3.3709649900809507D-02
  -4.1450735735657837D-03  -2.1670000000000000D-04

   5.9198513697197165D-02   1.1126346230843845D+00   1.4964588862363718D+00   1.9861905004340573D+00
   2.8059330025953902D-01   6.6710965564256879D-01  -1.7290776005853049D-01   9.0587634534875669D-02
  -1.7794617023974794D-02   1.8147985954379085D-03
METHOD IS STABLE                                                                                STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                       9.9999999999994895D-01   0.
  -2.9999999998241386D-01   0.                      -2.6963820335947381D-01  -1.3151136562698875D-01
  -2.9344427477178474D-01  -6.2373532877824302D-02  -2.9344427477178528D-01   6.2373532877824910D-02
  -2.6963820335936174D-01   1.3151136562417963D-01  -2.2981331889874225D-01   1.9283629963454753D-01
  -2.2981331889874225D-01  -1.9283629963454753D-01
   12-TH TRUNCATION-COEFFICIENT =    1.617571450-03
   13-TH TRUNCATION-COEFFICIENT =    7.929619590-03
   14-TH TRUNCATION-COEFFICIENT =    2.004727360-02

THETA =   -0.                        ,  B-SUB-THETA =   -0.
```

METHOD RW(.4)

ORDER 10

```
INPUT

STEP NUMBER  K =        9
METHOD CLASS  J =       2
NUMEER OF PREDETERMINE PARAMETERS  NP =      -6
KR=-INPUT NP =     6.   NP IS RESET TO K-2 =     7
    1-TH PRELIMINARY ROOT IS     1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS     1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    -4.0000000000000000D-01  -0.
    4-TH PRELIMINARY ROOT IS     4.0000000000000000D-01   1.4000000000000000D+02
    5-TH PRELIMINARY ROOT IS     4.0000000000000000D-01   1.5400000000000000D+02
    6-TH PRELIMINARY ROOT IS     4.0000000000000000D-01   1.6800000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                     TO   -0.                   BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                     TO   -0.                   BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -5.1438879207999105D-01   1.2201588484930801D+00   1.2962821466811967D+00
   8.1387716449284102D-02  -5.4106490559720499D-01  -3.8887075609458218D-01  -1.2989531662088351D-01
  -2.2470541230899108D-02  -1.6384000000000000D-03

   5.8075676259584623D-02   1.1609895648455116D+00   2.1488077944314700D+00   3.1036407613574835D+00
   1.3373829635994274D+00   1.1464178232403055D+00   3.2796596680774127D-03   1.1263482665174055D-01
  -1.3630524281157196D-02   1.7394678896933683D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                          9.9999999999791202D-01   0.
  -3.9999999999999778D-01   0.                         -3.5951760452072242D-01  -1.7534848752219282D-01
  -3.9125903302904632D-01  -8.3164710503765736D-02   -3.9125903302904704D-01   8.3164710503766546D-02
  -3.5951760447914898D-01   1.7534846749890618D-01   -3.0641775853165634D-01   2.5711506617939671D-01
  -3.0641775853165634D-01  -2.5711506617939671D-01
   12-TH TRUNCATION-COEFFICIENT =    1.50206924D-03
   13-TH TRUNCATION-COEFFICIENT =    7.36986622D-03
   14-TH TRUNCATION-COEFFICIENT =    1.86497309D-02

THETA =    -0.                 ,  B-SUB-THETA =    -0.
```

STABLE

METHOD W(.6)

ORDER 11

2-20

```
INPUT

STEP NUMBER  K =     10
METHOD CLASS  J =     2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -6
KR=-INPUT NP =    6.  NP IS RESET TO K-2 =    8
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    6.0000000000000000D-01   9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    6.0000000000000000D-01   1.1500000000000000D+02
    5-TH PRELIMINARY ROOT IS    6.0000000000000000D-01   1.4000000000000000D+02
    6-TH PRELIMINARY ROOT IS    6.0000000000000000D-01   1.6500000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                    TO   -0.                 BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                    TO   -0.                 BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -5.8550603340215139D-01   6.1147017678227717D-01   1.2008633155767801D+00
   6.0221684163611496D-01   7.4648634695634955D-02  -3.6543236905206721D-01  -3.9777330678414484D-01
  -2.3685404995603297D-01  -8.7037049494410775D-02  -1.6796160000000000D-02

   5.6549117558316730D-02   1.1794223191064626D+00   2.2057475894223616D+00   4.0516527290441940D+00
   2.4716910049620856D+00   3.0656424788798151D+00   6.9608256015789246D-01   7.6011816419234516D-01
   5.2011501940103609D-02   4.1684275195777039D-02  -6.4990187324993289D-04
METHOD IS STABLE
```

STABLE

```
ROOTS ARE

   1.0000000000000000D+00   0.                         9.9999999999981660D-01   0.
  -5.7955548241188269D-01  -1.5529147681525080D-01  -4.5962663779748364D-01   3.8567259926909519D-01
  -4.5962663779748440D-01  -3.8567259926909491D-01  -2.5357092452939385D-01  -5.4378468736434564D-01
   2.8076546628253434D-08  -6.0000000000040495D-01  -2.5357092452964123D-01   5.4378468738387888D-01
   2.6076937971553629D-08   5.9999999999999934D-01  -5.7955548245088733D-01   1.5529147678195099D-01
   13-TH TRUNCATION-COEFFICIENT =    1.351336640-03
   14-TH TRUNCATION-COEFFICIENT =    7.325603220-03
   15-TH TRUNCATION-COEFFICIENT =    2.042116340-02

THETA =    -0.                   ,  B-SUB-THETA =    -0.
```

METHOD W(.7)

ORDER 11

```
INPUT

STEP NUMBER  K =     10
METHOD CLASS  J =     2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -6
KR=-INPUT NP =     6.   NP IS RESET TO K-2 =     8
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    7.0000000000000000D-01   9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    7.0000000000000000D-01   1.1500000000000000D+02
    5-TH PRELIMINARY ROOT IS    7.0000000000000000D-01   1.4000000000000000D+02
    6-TH PRELIMINARY ROOT IS    7.0000000000000000D-01   1.6500000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                        TO   -0.                      BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                        TO   -0.                      BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -1.0164237056358433D+00   1.6791539429726283D-01   1.3812132456494113D+00
   1.4716888279333875D+00   6.5024828766999571D-01  -2.4471899668461267D-01  -6.2153414849839357D-01
  -5.1115772218685649D-01  -2.3956321254435133D-01  -5.7648010000000000D-02

   5.5882250823470028D-02   1.2108770093303297D+00   2.6897470258232360D+00   5.2993162394060103D+00
   4.3946729968204630D+00   4.9672265799196244D+00   2.0811361030535331D+00   1.4877133820205324D+00
   3.0439162440683544D-01   9.7874788676236224D-02   1.5648157380348494D-03
METHOD IS STABLE
```

STABLE

2-21

```
ROOTS ARE

   1.0000000000000000D+00   0.                         9.9999999999998850D-01   0.
  -6.7614806285932300D-01  -1.8117338957963929D-01  -5.3623107743039758D-01   4.4995136581394439D-01
  -5.3623107743039847D-01  -4.4995136581394406D-01  -2.9583274528429283D-01  -6.3441546861506991D-01
   3.2755973399631172D-08  -7.0000000000047245D-01  -2.9583274528458143D-01   6.3441546861452536D-01
   3.2756427633479229D-08   6.9999999999999923D-01  -6.7614806265936856D-01   1.8117338957894282D-01
   13-TH TRUNCATION-COEFFICIENT =    1.25251976D-03
   14-TH TRUNCATION-COEFFICIENT =    6.82915220D-03
   15-TH TRUNCATION-COEFFICIENT =    1.91258113D-02

THETA =    -0.                  ,  B-SUB-THETA =   -0.
```

METHOD W(.8)
ORDER 11


```
INPUT

STEP NUMBER  K =     10
METHOD CLASS  J =     2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -6
KR=-INPUT NP =    6.  NP IS RESET TO K-2 =    8
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    8.0000000000000000D-01   1.1500000000000000D+02
    5-TH PRELIMINARY ROOT IS    8.0000000000000000D-01   1.4000000000000000D+02
    6-TH PRELIMINARY ROOT IS    8.0000000000000000D-01   1.6500000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                          TO   -0.                     BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                          TO   -0.                     BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -1.4473413778695352D+00  -4.3339171541119737D-01   1.3090400241284943D+00
   2.1907253822836663D+00   1.6056516882460461D+00   2.3898792103298006D-01  -7.7559247718753595D-01
  -9.5235174706268661D-01  -5.6815553816023143D-01  -1.6777216000000000D-01

   5.5158510762583314D-02   1.2426899634004317D+00   3.1845374364012849D+00   6.7848809106722695D+00
   6.9287330729057744D+00   7.9295303100891434D+00   4.4579802306446788D+00   2.9704195132262392D+00
   6.6837583934954800D-01   2.4530633533413359D-01   7.5089710432852117D-03
METHOD IS STABLE                                                                                        STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                        9.9999999999999684D-01   0.
  -7.7274064326784895D-01  -2.0705530237594709D-01  -6.1283551706331152D-01   5.1423013235879359D-01
  -6.1283551706331253D-01  -5.1423013235879321D-01  -3.3809456603919180D-01  -7.2504624984579419D-01
   3.7435398171007036D-08  -8.0000000000053994D-01  -3.3809456603952163D-01   7.2504624984517185D-01
   3.7435917295404829D-08   7.9999999999999912D-01  -7.7274064326784976D-01   2.0705530237593465D-01
   13-TH TRUNCATION-COEFFICIENT =    1.07936070D-03
   14-TH TRUNCATION-COEFFICIENT =    5.99539692D-03
   15-TH TRUNCATION-COEFFICIENT =    1.70346441D-02

THETA =    -0.                         ,  B-SUB-THETA =    -0.
```

METHOD RW(.2)

ORDER 11

```
INPUT

STEP NUMBER  K =     10
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -6
KR=-INPUT NP =     6.  NP IS RESET TO K-2 =     8
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.4000000000000000D+02
    4-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.5000000000000000D+02
    5-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.6000000000000000D+02
    6-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.7000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                       TO   -0.                     BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                       TO   -0.                     BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   5.7737196355434354D-01   9.2849619967883761D-01   6.1802933031588966D-02
  -3.0443982399454021D-01  -1.9133731867066289D-01  -5.9397218318798793D-02  -1.1120967677960806D-02
  -1.2872794084749165D-03  -8.5928194332522013D-05  -2.5600000000000000D-06

   5.8372933286784499D-02   1.0930731163290584D+00   9.6056186551335372D-01   1.6021144522947308D+00
  -5.2830351318366102D-01   9.7005512378993554D-01  -5.8390679100127741D-01   2.9694806146339207D-01
  -9.6408521995155142D-02   1.9064910976066356D-02  -1.7134711205418965D-03
METHOD IS STABLE                                                                                    STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                       9.9999999999999115D-01   0.
  -1.9696154753267585D-01  -3.4729652942888353D-02  -1.8793851846658633D-01  -6.8404044299911060D-02
  -1.7320507295773777D-01  -1.0000001350852278D-01  -1.5320887926582817D-01  -1.2855753308969835D-01
  -1.9696154753268124D-01   3.4729652942896154D-02  -1.8793851846415830D-01   6.8404044265572293D-02
  -1.7320507295773777D-01   1.0000001350852278D-01  -1.5320887926582817D-01   1.2855753308969835D-01
   13-TH TRUNCATION-COEFFICIENT =    1.53959028D-03
   14-TH TRUNCATION-COEFFICIENT =    8.32094895D-03
   15-TH TRUNCATION-COEFFICIENT =    2.31328947D-02

THETA =    -0.                  ,  B-SUB-THETA =    -0.
```

METHOD RW(.3)

ORDER 11

INPUT

```
STEP NUMBER  K =     10
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -6
KR=-INPUT NP =     6.  NP IS RESET TO K-2 =     8
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.4000000000000000D+02
    4-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.5000000000000000D+02
    5-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.6000000000000000D+02
    6-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.7000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                     TO   -0.                    BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                     TO   -0.                    BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
```



```
COEFFICIENTS ARE

  -1.0000000000000000D+00  -1.3394205466848469D-01   1.2051743946088999D+00   8.1330275258914925D-01
  -1.4101408776077797D-01  -4.1490310333693353D-01  -2.3929013412328490D-01  -7.4175462377124046D-02
  -1.3662271173590712D-02  -1.4244237578533253D-03  -6.5610000000000000D-05

   5.7182508527823879D-02   1.1474569251392122D+00   1.6830081766419943D+00   2.6144608157373375D+00
   3.7160917385155789D-01   1.2625417988303139D+00  -4.2447159383537625D-01   2.9192515182241374D-01
  -8.5918510037922645D-02   1.7619236219773225D+02  -1.5687554273339018D-03
METHOD IS STABLE
```

STABLE

```
ROOTS ARE

   1.0000000000000000D+00   0.                          9.9999999999958259D-01   0.
  -2.9544232129901377D-01  -5.2094479414332542D-02  -2.8190777769987950D-01  -1.0260606644986659D-01
  -2.5980760943660665D-01  -1.5000002026278416D-01  -2.2981331889874225D-01  -1.9283629963454753D-01
  -2.9544232129902187D-01   5.2094479414344230D-02  -2.8190777769623746D-01   1.0260606642835844D-01
  -2.5980760943660665D-01   1.5000002026278417D-01  -2.2981331889874225D-01   1.9283629963454753D-01
   13-TH TRUNCATION-COEFFICIENT =    1.42054496D-03
   14-TH TRUNCATION-COEFFICIENT =    7.66455722D-03
   15-TH TRUNCATION-COEFFICIENT =    2.13842672D-02

THETA =   -0.                       ,  B-SUB-THETA =   -0.
```

METHOD RW(.4)

ORDER 11


```
INPUT

STEP NUMBER  K =     10
METHOD CLASS  J =     2
NUMBER OF PREDETERMINE PARAMETERS  NP =    -6
KR=-INPUT NP =    6.  NP IS RESET TO K-2 =    8
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.4000000000000000D+02
    4-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.5000000000000000D+02
    5-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.6000000000000000D+02
    6-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.7000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                  BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                  BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

 -1.0000000000000000D+00  -8.4525607289131292D-01   1.0234726529327246D+00   1.6961126972268482D+00
  5.4466086347377757D-01  -4.9210370039549612D-01  -5.7392901687043617D-01  -2.7073538133371049D-01
 -7.1223233267831336D-02  -1.0343448874562818D-02  -6.5536000000000000D-04

  5.6081231678862900D-02   1.2001032217466760D+00   2.4452080197246784D+00   4.0818836190332743D+00
  1.9976136842547980D+00   2.1447278789967009D+00  -8.9875053450542842D-03   3.6352218298054012D-01
 -6.4427446724450660D-02   1.7439631376073531D-02  -1.4192871750566732D-03
METHOD IS STABLE                                                                                    STABLE

ROOTS ARE

  1.0000000000000000D+00   0.                       9.9999999998207515D-01   0.
 -3.9392309506540036D-01  -6.9459305886331967D-02  -3.7587703693317266D-01  -1.3680808859982212D-01
 -3.4641014591547553D-01  -2.0000002701704555D-01  -3.0641775853165634D-01  -2.5711506617939671D-01
 -3.9392309506536249D-01   6.9459305885792307D-02  -3.7587703692831661D-01   1.3680808857114459D-01
 -3.4641014591547553D-01   2.0000002701704555D-01  -3.0641775853165634D-01   2.5711506617939671D-01
  13-TH TRUNCATION-COEFFICIENT =   1.31566469D-03
  14-TH TRUNCATION-COEFFICIENT =   7.12335273D-03
  15-TH TRUNCATION-COEFFICIENT =   1.98405358D-02

THETA =    -0.                     ,  B-SUB-THETA =    -0.
RUN COMPLETED
```

METHOD W(.6)

ORDER 12

INPUT

STEP NUMBER K = 11
METHOD CLASS J = 2
NUMBER OF PREDETERMINE PARAMETERS NP = -7
KR=-INPUT NP = 7. NP IS RESET TO K-2 = 9

1-TH PRELIMINARY ROOT IS	1.00000000000000000D+00	-0.
2-TH PRELIMINARY ROOT IS	1.00000000000000000D+00	-0.
3-TH PRELIMINARY ROOT IS	-6.00000000000000000D-01	-0.
4-TH PRELIMINARY ROOT IS	6.00000000000000000D-01	9.00000000000000000D+01
5-TH PRELIMINARY ROOT IS	6.00000000000000000D-01	1.15000000000000000D+02
6-TH PRELIMINARY ROOT IS	6.00000000000000000D-01	1.40000000000000000D+02
7-TH PRELIMINARY ROOT IS	6.00000000000000000D-01	1.65000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS NA = -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM -0. TO -0. BY -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM -0. TO -0. BY -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

-1.00000000000000000D+00	-1.18550603340215140D+00	2.60166556740986340D-01	1.56774542164614640D+00
1.52273483098218300D+00	5.56178739677303930D-01	-3.20523188234686230D-01	-6.17032728215385160D-01
-4.75518034028519870D-01	-2.29149479469230560D-01	-6.90183896966464650D-02	-1.00776960000000000D-02
5.43869789341376490D-02	1.23713531450742260D+00	2.79448335655638980D+00	5.73185415568715930D+00
4.18917689640950540D+00	5.54756512622780170D+00	1.53656000311504620D+00	1.89127344626617730D+00
1.51329527465962390D-01	1.91808800669688640D-01	5.77138378246402490D-04	1.77219750022912100D-03

METHOD IS STABLE

STABLE

ROOTS ARE

1.00000000000000000D+00	0.	9.99999999999982690D-01	0.
-5.99999999979544580D-01	0.	-5.79555482411882690D-01	-1.55291476815258800D-01
-4.59626637797483640D-01	3.85672599269095190D-01	-4.59626637797484400D-01	-3.85672599269094910D-01
-2.53570924529393850D-01	-5.43784687384345640D-01	2.80765486282524050D-08	-6.00000000000404950D-01
-2.53570924529641230D-01	5.43784687383878880D-01	2.80769379715535880D-08	5.99999999999999340D-01
-5.79555482450887330D-01	1.55291476781950990D-01		

14-TH TRUNCATION-COEFFICIENT = 1.18053937D-03
15-TH TRUNCATION-COEFFICIENT = 6.98180602D-03
16-TH TRUNCATION-COEFFICIENT = 2.11993778D-02

THETA = -0. , B-SUB-THETA = -0.

METHOD W(.7)

ORDER 12

```
INPUT

STEP NUMBER  K =     11
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -7
KR=-INPUT NP =     7.   NP IS RESET TO K-2 =     9
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    3-TH PRELIMINARY ROOT IS   -7.0000000000000000D-01   -0.
    4-TH PRELIMINARY ROOT IS    7.0000000000000000D-01    9.0000000000000000D+01
    5-TH PRELIMINARY ROOT IS    7.0000000000000000D-01    1.1500000000000000D+02
    6-TH PRELIMINARY ROOT IS    7.0000000000000000D-01    1.4000000000000000D+02
    7-TH PRELIMINARY ROOT IS    7.0000000000000000D-01    1.6500000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                       TO    -0.                      BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                       TO    -0.                      BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -1.7164237056356433D+00  -5.2358119964782747D-01   1.5127540216574953D+00
   2.4385380998879754D+00   1.6804304672233670D+00   2.1045484468438433D-01  -7.9283741817762243D-01
  -9.4623162613573198D-01  -5.9739361807515087D-01  -2.2535625878104593D-01  -4.0353607000000000D-02

   5.3752967229970020D-02   1.2734167044352588D+00   3.4202503347119664D+00   7.5334709504097760D+00
   7.4015307785496677D+00   9.0272266978909519D+00   4.5744656888002667D+00   3.6471722400130080D+00
   9.9445919889370690D-01   4.2605952340352144D-01   4.6655048282900124D-02   3.2246546101244020D-03
METHOD IS STABLE
                                                                                              STABLE
ROOTS ARE

   1.0300000000000000D+00   0.                        9.9999999999999522D-01   0.
  -6.9999999999927013D-01   0.                       -6.7614806285932300D-01  -1.8117338957963929D-01
  -5.3623107743039758D-01   4.4995136581394439D-01   -5.3623107743039847D-01  -4.4995136581394406D-01
  -2.9583274528429283D-01  -6.3441546861506991D-01    3.2755973399631102D-08  -7.0000000000047245D-01
  -2.9583274528458143D-01   6.3441546861452536D-01    3.2756427633479194D-08   6.9999999999999923D-01
  -6.7614806285936856D-01   1.8117338957894282D-01
    14-TH TRUNCATION-COEFFICIENT =    1.15101875D-03
    15-TH TRUNCATION-COEFFICIENT =    6.78795525D-03
    16-TH TRUNCATION-COEFFICIENT =    2.05657434D-02

THETA =    -0.                        ,  B-SUB-THETA =    -0.
```

METHOD W(.8)

ORDER 12


```
INPUT

STEP NUMBER  K =     11
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -7
KR=-INPUT NP =     7.  NP IS RESET TO K-2 =      9
    1-TH PRELIMINARY ROOT IS    1.00000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.00000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS   -8.00000000000000000D-01  -0.
    4-TH PRELIMINARY ROOT IS    8.00000000000000000D-01   9.00000000000000000D+01
    5-TH PRELIMINARY ROOT IS    8.00000000000000000D-01   1.15000000000000000D+02
    6-TH PRELIMINARY ROOT IS    8.00000000000000000D-01   1.40000000000000000D+02
    7-TH PRELIMINARY ROOT IS    8.00000000000000000D-01   1.65000000000000000D+02
NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                          TO   -0.                       BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                          TO   -0.                       BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.00000000000000000D+00  -2.2473413778695352D+00  -1.5912648177068255D+00   9.6232665179953639D-01
   3.2379574015864617D+00   3.3564319940729791D+00   1.5236692716298169D+00  -5.8440214036115190D-01
  -1.5728257288127156D+00  -1.3300369358103809D+00  -6.2229659052818515D-01  -1.3421772800000000D-01

   5.3215661501514806D-02   1.3081881338822519D+00   4.0718327137628624D+00   9.6530809878696008D+00
   1.1715497545291983D+01   1.4370213127028212D+01   9.9040881201023439D+00   7.1779439538945891D+00
   2.9241413718542360D+00   1.0468637561725398D+00   1.8238269743883852D-01   7.9500260956966749D-03
METHOD IS STABLE
                                                                                                  STABLE
ROOTS ARE

   1.00000000000000000D+00   0.                          9.9999999999982209D-01   0.
  -7.9999999989613931D-01   0.                         -7.7274064326784895D-01  -2.0705530237594709D-01
  -6.1283551706331152D-01   5.1423013235879359D-01  -6.1283551706331253D-01  -5.1423013235879321D-01
  -3.3809456603919180D-01  -7.2504624984579419D-01   3.7435398171001974D-08  -8.0000000000053994D-01
  -3.3809456603952163D-01   7.2504624984517185D-01   3.7435917295404801D-08   7.9999999999999912D-01
  -7.7274064326784978D-01   2.0705530237593465D-01
   14-TH TRUNCATION-COEFFICIENT =    1.18540424D-03
   15-TH TRUNCATION-COEFFICIENT =    6.92136906D-03
   16-TH TRUNCATION-COEFFICIENT =    2.08011550D-02

THETA =    -0.                      ,  B-SUB-THETA =    -0.
```

```
INPUT

STEP NUMBER  K =     11
METHOD CLASS  J =     2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -7
KR=-INPUT NP =     7.  NP IS RESET TO K-2 =     9
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS   -2.0000000000000000D-01  -0.
    4-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.4000000000000000D+02
    5-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.5000000000000000D+02
    6-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.6000000000000000D+02
    7-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.7000000000000000D+02
NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                        TO    -0.                    BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                        TO    -0.                    BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   3.7737196355434354D-01   1.0439705923897063D+00   2.4750217296735651D-01
  -2.9207923738822242D-01  -2.5222528346957093D-01  -9.7654662052931370D-02  -2.3000411341720566D-02
  -3.5114729440670780D-03  -3.4336407602750531D-04  -1.9745638866504403D-05  -5.1200000000000000D-07

   5.6525424946591069D-02   1.1250702947285430D+00   1.0775635300685268D+00   2.0990657015293254D+00
  -8.1755837498854510D-01   1.7179432743225679D+00  -1.2434446194126549D+00   7.8984445552696842D-01
  -3.4185778583439364D-01   1.0139616528767377D-01  -1.8223080667456353D-02   1.5046141160850505D-03
METHOD IS STABLE
                                                                                              STABLE
ROOTS ARE

   1.0000000000000000D+00   0.                           9.9999999999998389D-01   0.
  -1.9999999999966821D-01   0.                          -1.9696154753267585D-01  -3.4729652942888353D-02
  -1.8793851846658633D-01  -6.8404044299911060D-02      -1.7320507295773777D-01  -1.0000001350852278D-01
  -1.5320887926582817D-01  -1.2855753308969835D-01      -1.9696154753268124D-01   3.4729652942896154D-02
  -1.8793851846415830D-01   6.8404044285572293D-02      -1.7320507295773777D-01   1.0000001350852278D-01
  -1.5320887926582817D-01   1.2855753308969835D-01
   14-TH TRUNCATION-COEFFICIENT =    1.36343316D-03
   15-TH TRUNCATION-COEFFICIENT =    8.05987938D-03
   16-TH TRUNCATION-COEFFICIENT =    2.44555759D-02

THETA =    -0.                        ,  B-SUB-THETA =    -0.
```

METHOD RW(.3)

ORDER 12

```
INPUT
STEP NUMBER  K =      11
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -7
KR=-INPUT NP =    7.  NP IS RESET TO K-2 =     9
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS   -3.0000000000000000D-01  -0.
    4-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.4000000000000000D+02
    5-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.5000000000000000D+02
    6-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.6000000000000000D+02
    7-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.7000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                     BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                     BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -4.3394205466848469D-01   1.1649917782083545D+00   1.1748550709718192D+00
   1.0297673801596680D-01  -4.5720732966516692D-01  -3.6376106512436496D-01  -1.4596250261410952D-01
  -3.5914909886727926D-02  -5.5231051099305389D-03  -4.9293712735599760D-04  -1.9683000000000000D-05

   5.5335800081064238D-02   1.1849254706119154D+00   1.9256762896119777D+00   3.4240701624452765D+00
   5.4653363114207766D-01   2.2272038533887354D+00  -8.9888835658923614D-01   7.7399746110248235D-01
  -3.0304785820653947D-01   9.3412647780176619D-02  -1.6596777475757984D-02   1.3760818185594703D-03
METHOD IS STABLE                                                                                    STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                         9.9999999999684870D-01   0.
  -2.9999999996513810D-01   0.                        -2.9544232129901377D-01  -5.2094479414332542D-02
  -2.8190777769987950D-01  -1.0260606644986659D-01  -2.5980760943660665D-01  -1.5000002026278416D-01
  -2.2981331889874225D-01  -1.9283629963454753D-01  -2.9544232129902187D-01   5.2094479414344231D-02
  -2.8190777769623746D-01   1.0260606642835844D-01  -2.5980760943660665D-01   1.5000002026278417D-01
  -2.2981331889874225D-01   1.9283629963454753D-01
   14-TH TRUNCATION-COEFFICIENT =   1.25357290D-03
   15-TH TRUNCATION-COEFFICIENT =   7.41650378D-03
   16-TH TRUNCATION-COEFFICIENT =   2.25228316D-02

THETA =    -0.                      ,  B-SUB-THETA =    -0.
```

METHOD RW(.4)

ORDER 12

```
INPUT

STEP NUMBER  K =     11
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =    -7
KR=-INPUT NP =    7.  NP IS RESET TO K-2 =    9
    1-TH PRELIMINARY ROOT IS    1.00000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.00000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS   -4.00000000000000000D-01  -0.
    4-TH PRELIMINARY ROOT IS    4.00000000000000000D-01   1.40000000000000000D+02
    5-TH PRELIMINARY ROOT IS    4.00000000000000000D-01   1.50000000000000000D+02
    6-TH PRELIMINARY ROOT IS    4.00000000000000000D-01   1.60000000000000000D+02
    7-TH PRELIMINARY ROOT IS    4.00000000000000000D-01   1.70000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                     TO   -0.                  BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                     TO   -0.                  BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.00000000000000000D+00  -1.24525607289131290D+00   6.85370223776199440D-01   2.10550175839993800D+00
   1.22310594236451680D+00  -2.74239355005985090D-01  -7.70770497028634620D-01  -5.00306988081884960D-01
  -1.79517385801316030D-01  -3.88327421816955520D-02  -4.79273954982512710D-03  -2.62144000000000000D-04

   5.42393011186507500D-02   1.24279695058055480D+00   2.82394312761168060D+00   5.36388536935815030D+00
   3.02253004699809850D+00   3.79474527151663310D+00  -2.06827256438686990D-03   9.67764265712527650D-01
  -2.22937115967239240D-01   9.29748334979614740D-02  -1.47046707869611020D-02   1.27421569018940000D-03
METHOD IS STABLE                                                                                        STABLE

ROOTS ARE

   1.00000000000000000D+00   0.                          9.99999999862623640D-01   0.
  -3.99999999999745250D-01   0.                         -3.93923095065400360D-01  -6.94593058863319680D-02
  -3.75877036933172660D-01  -1.36808088599822120D-01  -3.46410145915475530D-01  -2.00000027017045550D-01
  -3.06417758531656340D-01  -2.57115066179396710D-01  -3.93923095065362490D-01   6.94593058857923080D-02
  -3.75877036928316610D-01   1.36808088571144590D-01  -3.46410145915475530D-01   2.00000027017045550D-01
  -3.06417758531656340D-01   2.57115066179396710D-01
   14-TH TRUNCATION-COEFFICIENT =   1.15774044D-03
   15-TH TRUNCATION-COEFFICIENT =   6.85451729D-03
   16-TH TRUNCATION-COEFFICIENT =   2.08322459D-02

THETA =   -0.                  ,  B-SUB-THETA =   -0.
RUN COMPLETED
```

METHOD W(.1)

ORDER 13

```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS   NP =     -7
KR=-INPUT NP =     7.   NP IS RESET TO K-2 =    10
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    3-TH PRELIMINARY ROOT IS    1.0000000000000000D-01    9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    1.0000000000000000D-01    1.0800000000000000D+02
    5-TH PRELIMINARY ROOT IS    1.0000000000000000D-01    1.2600000000000000D+02
    6-TH PRELIMINARY ROOT IS    1.0000000000000000D-01    1.4400000000000000D+02
    7-TH PRELIMINARY ROOT IS    1.0000000000000000D-01    1.6200000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                    TO    -0.                    BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                    TO    -0.                    BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000+00   1.4686248931802437D+00  -8.8429542754937622D-02  -2.5834827901316317D-01
  -9.6773385766729457D-02  -2.1295064984709046D-02  -3.3408319778838792D-03  -3.9813195794981781D-04
  -3.6874636128871680D-05  -2.6362227365100291D-06  -1.4065225425805510D-07  -5.1137510681975630D-09
  -1.0000000000000000D-10

   5.6762157308020303D-02   1.0593957987015829D+00  -7.9443561606243984D-02   1.5783758568175684D+00
  -2.3190167562916037D+00   3.0003309697766689D+00  -2.3938471550784841D+00   2.1169189464277423D+00
  -1.1533399144266953D+00   4.5430140639830321D-01  -1.2235899623334969D-01   2.0182908826656998D-02
  -1.5390525551943760D-03
METHOD IS STABLE
```

STABLE

```
ROOTS ARE

   1.0000076320393901D+00   0.                         9.9999236741012959D-01   0.
  -1.3049987174167449D-01   0.                         1.1582037465637637D-02   9.3652862601080468D-02
  -1.9865014471142459D-02   1.1702786297966354D-01  -7.4535430928684090D-02   1.0792677807674611D-01
  -1.1761920933163753D-01   6.5719970612848790D-02  -1.1761920933165350D-01  -6.5719970612840596D-02
  -7.4535430928684934D-02  -1.0792677807674624D-01  -1.9865014471138300D-02  -1.1702786297964600D-01
   1.1582037467727797D-02  -9.3652862598926602D-02
   15-TH TRUNCATION-COEFFICIENT =   1.39144668D-03
   16-TH TRUNCATION-COEFFICIENT =   8.91883783D-03
   17-TH TRUNCATION-COEFFICIENT =   2.92868347D-02

THETA =    -0.                 ,  B-SUB-THETA =    -0.
```

2-32

METHOD W(.2)

ORDER 13

```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =     2
NUMBER OF PREDETERMINE PARAMETERS  NP =    -7
KR=-INPUT NP =    7.  NP IS RESET TO K-2 =   10
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.0800000000000000D+02
    5-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.2600000000000000D+02
    6-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.4400000000000000D+02
    7-TH PRELIMINARY ROOT IS    2.0000000000000000D-01   1.6200000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                        TO   -0.                  BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                        TO   -0.                  BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   9.3724978636048739D-01   5.2078140170122430D-01  -8.7973642342369348D-02
  -2.0354005525118570D-01  -1.1374561871591760D-01  -4.0125513689788261D-02  -1.0296056854612244D-02
  -2.0135448577019883D-03  -3.0077497304649233D-04  -3.3363136543144954D-05  -2.5158405469171523D-06
  -1.0240000000000000D-07

   5.5973189143654597D-02   1.0995567414045301D+00   4.3496771279266143D-01   1.9256539819055605D+00
  -1.9519715992408883D+00   2.8855856536272597D+00  -2.7002934799959762D+00   1.9941860184906005D+00
  -1.0851011373301666D+00   4.2796250197912793D-01  -1.1530781706591294D-01   1.9029411310558557D-02
  -1.4516079453029917D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                        9.9999999999982077D-01   0.
  -1.1755703985829176D-01   1.6180340657648722D-01  -1.6180339007330471D-01   1.1755706257320476D-01
  -1.9021129805327946D-01   6.1803414896642152D-02  -1.6180339007327699D-01  -1.1755706257301161D-01
  -1.1755703985847311D-01  -1.6180340657649600D-01  -6.1803388193867724D-02  -1.9021130672951819D-01
   9.3588907904895293D-09  -2.0000000000013263D-01  -1.9021129805327944D-01  -6.1803414896642135D-02
  -6.1803388193887483D-02   1.9021130672953073D-01   9.3589793238461194D-09   1.9999999999999978D-01
   15-TH TRUNCATION-COEFFICIENT =    1.31749992D-03
   16-TH TRUNCATION-COEFFICIENT =    8.44899916D-03
   17-TH TRUNCATION-COEFFICIENT =    2.77583089D-02

THETA =   -0.                , B-SUB-THETA =   -0.
```

STABLE

METHOD W(.3)

ORDER 13

```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =     2
NUMBER OF FREDETERMINE PARAMETERS   NP =     -7
KR=-INPUT NP =     7.   NP IS RESET TO K-2 =   10
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    3-TH PRELIMINARY ROOT IS    3.0000000000000000D-01    9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    3.0000000000000000D-01    1.0800000000000000D+02
    5-TH PRELIMINARY ROOT IS    3.0000000000000000D-01    1.2600000000000000D+02
    6-TH PRELIMINARY ROOT IS    3.0000000000000000D-01    1.4400000000000000D+02
    7-TH PRELIMINARY ROOT IS    3.0000000000000000D-01    1.6200000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                           TO    -0.                       BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                           TO    -0.                       BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   4.0587467954073109D-01   8.2763283336848577D-01   3.3512780011853754D-01
  -1.2163176479386019D-01  -2.1875662025953066D-01  -1.4256301482628000D-01  -6.1050866187757860D-02
  -1.9709379136399074D-02  -4.5358680044979676D-03  -7.8861415715332267D-04  -9.2780762275332633D-05
  -5.9049000000000000D-06

   5.5231103580133255D-02   1.1386742246551796D+00   9.7555678193391964D-01   2.5604374374914108D+00
  -1.2657422231314579D+00   3.0061412580560939D+00  -2.4335001038157806D+00   1.9138904696480053D+00
  -1.0170677573977989D+00   4.0558170805438029D-01  -1.0683369306744333D-01   1.8005752392703642D-02
  -1.3728220497176605D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                        9.9999999999979468D-01   0.
  -2.8531694707991917D-01  -9.2705122344963233D-02  -2.4270508510993699D-01  -1.7633559385950209D-01
  -1.7633555978743869D-01  -2.4270510986472937D-01  -9.2705082290746542D-02  -2.8531696009421216D-01
   1.4038981736487648D-08  -3.0000000000075915D-01  -2.4270508510989117D-01   1.7633559385952406D-01
  -1.7633555978743815D-01   2.4270510986473035D-01  -9.2705082291184806D-02   2.8531696009641437D-01
   1.4038470754061132D-08   3.0000000000000040D-01  -2.8531694707991917D-01   9.2705122344963232D-02
   15-TH TRUNCATION-COEFFICIENT =    1.25063920D-03
   16-TH TRUNCATION-COEFFICIENT =    8.02383940D-03
   17-TH TRUNCATION-COEFFICIENT =    2.63739680D-02

THETA =    -0.                        ,  B-SUB-THETA =    -0.
```

STABLE

METHOD W(.4)

ORDER 13

```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =   -7
KR=-INPUT NP =    7.  NP IS RESET TO K-2 =   10
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.0800000000000000D+02
    5-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.2600000000000000D+02
    6-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.4400000000000000D+02
    7-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.6200000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                       TO    -0.                    BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                       TO    -0.                    BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

 -1.0000000000000000D+00  -1.2550042727902521D-01   8.3212475224684679D-01   8.3495993847571358D-01
  2.4540374517930276D-01  -1.8859194279601895D-01  -2.8093691424798024D-01  -1.9106920981467814D-01
 -8.8238987204214955D-02  -2.9636206004900515D-02  -7.2266381950235262D-03  -1.1832527600215820D-03
 -1.0485760000000000D-04

  5.4531052204463608D-02   1.1768371194978018D+00   1.5413646271108292D+00   3.4990651923228603D+00
 -9.1673094174821697D-02   3.6015905250406048D+00  -1.9055403688986034D+00   1.9671155012322385D+00
 -9.1528738030453831D-01   3.9521857781446208D-01  -1.0138384232917249D-01   1.7252309149935842D-02
 -1.2974295560471875D-03
METHOD IS STABLE

ROOTS ARE

  1.0000000000000000D+00   0.                        9.9999999997688558D-01   0.
 -3.8042259610658884D-01  -1.2360682979320571D-01  -3.2360678014658265D-01  -2.3511412514600278D-01
 -2.3511407971658492D-01  -3.2360681315297250D-01  -1.2360677638766206D-01  -3.8042261345894955D-01
  1.8718642315317619D-08  -4.0000000000101220D-01  -3.2360678014652156D-01   2.3511412514603208D-01
 -2.3511407971658420D-01   3.2360681315297380D-01  -1.2360677638824641D-01   3.8042261346188582D-01
  1.8717961005414757D-08   4.0000000000000053D-01  -3.8042259610655890D-01   1.2360682979328431D-01
  15-TH TRUNCATION-COEFFICIENT =     1.189850830-03
  16-TH TRUNCATION-COEFFICIENT =     7.637045380-03
  17-TH TRUNCATION-COEFFICIENT =     2.511367360-02

THETA =    -0.                  ,  B-SUB-THETA =    -0.
```

STABLE

METHOD W(.5)

ORDER 13

```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -7
KR=-INPUT NP =     7.  NP IS RESET TO K-2 =    10
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    5.0000000000000000D-01   9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    5.0000000000000000D-01   1.0800000000000000D+02
    5-TH PRELIMINARY ROOT IS    5.0000000000000000D-01   1.2600000000000000D+02
    6-TH PRELIMINARY ROOT IS    5.0000000000000000D-01   1.4400000000000000D+02
    7-TH PRELIMINARY ROOT IS    5.0000000000000000D-01   1.6200000000000000D+02
NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                          TO   -0.                   BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                          TO   -0.                   BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -6.5687553409878151D-01   5.3425715833630735D-01   1.2355266628353148D+00
   8.9180275015690594D-01   1.5598018523684531D-01  -3.2497650576418257D-01  -3.9940767173701620D-01
  -2.6620003588269734D-01  -1.2143083679618506D-01  -3.9274314731435389D-02  -8.4252950550733653D-03
  -9.7656250000000000D-04

   5.3868420483189367D-02   1.2141286813189530D+00   2.1314982807961602D+00   4.7572835456149577D+00
   1.7483449567213572D+00   5.0116319924792067D+00  -7.6585127361985423D-01   2.3860127977262824D+00
  -6.7068962215586953D-01   4.3307998153570843D-01  -8.4935240626390490D-02   1.7872425218471507D-02
  -1.1921073823767756D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                        9.9999999999934842D-01   0.
  -4.7552824513319863D-01  -1.5450853724160539D-01  -4.0450847518322831D-01  -2.9389265643250348D-01
  -2.9389259964573115D-01  -4.0450851644121562D-01  -1.5450847048457757D-01  -4.7552826682368694D-01
   2.3398302894145911D-08  -5.0000000000126525D-01  -4.0450847518315194D-01   2.9389265643254010D-01
  -2.9389259964573025D-01   4.0450851644121725D-01  -1.5450847048530801D-01   4.7552826682735728D-01
   2.3397451256768370D-08   5.0000000000000066D-01  -4.7552824513319862D-01   1.5450853724160539D-01
   15-TH TRUNCATION-COEFFICIENT =   1.13386702D-03
   16-TH TRUNCATION-COEFFICIENT =   7.28100220D-03
   17-TH TRUNCATION-COEFFICIENT =   2.39538716D-02

THETA =   -0.                    ,  B-SUB-THETA =   -0.
```

2-36

W(.5)

STABLE

METHOD W(.6)

ORDER 13

2-37

```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -7
KR=-INPUT NP =    7.  NP IS RESET TO K-2 =   10
    1-TH PRELIMINARY ROOT IS    1.00000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.00000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    6.00000000000000000D-01   9.00000000000000000D+01
    4-TH PRELIMINARY ROOT IS    6.00000000000000000D-01   1.08000000000000000D+02
    5-TH PRELIMINARY ROOT IS    6.00000000000000000D-01   1.26000000000000000D+02
    6-TH PRELIMINARY ROOT IS    6.00000000000000000D-01   1.44000000000000000D+02
    7-TH PRELIMINARY ROOT IS    6.00000000000000000D-01   1.62000000000000000D+02
NUMBER OF  BITRARY PARAMETERS  NA =    -0
THETA IS AN RBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                     TO   -0.                    BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                     TO   -0.                    BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

 -1.00000000000000000D+00  -1.18325064091853780D+00  -6.59699483631325460D-02   1.36083186330349740D+00
  1.70958554154209930D+00   9.68041739001252710D-01  -3.58926991371071190D-02  -5.82394575127586440D-01
 -5.98747767695988240D-01  -3.66829942773246680D-01  -1.52869619546280300D-01  -4.14571326849703080D-02
 -6.04661760000000000D-03

  5.32368246071915320D-02   1.25065309163422640D+00   2.74496665411505340D+00   6.35088431955061080D+00
  4.44001044360136340D+00   7.68417958063971590D+00   1.55224834052290160D+00   3.64883565526550960D+00
 -1.08602591358597780D-02   6.34282003628512550D-01  -2.92745779399967960D-02   2.51231803338318860D-02
 -8.88598719089782340D-04
METHOD IS STABLE

ROOTS ARE

  1.00000000000000000D+00   0.                         9.99999999999977580D-01   0.
 -5.70633894159516390D-01   1.85410244688686720D-01  -4.85410170219873970D-01   3.52671187719004170D-01
 -3.52671119574877380D-01   4.85410219729458740D-01  -1.85410164581493080D-01   5.70633920188424330D-01
  2.80779634729754530D-08   6.00000000001518300D-01  -4.85410170219782330D-01  -3.52671187719048120D-01
 -3.52671119574876290D-01  -4.85410219729460700D-01  -1.85410164582369610D-01  -5.70633920192828730D-01
  2.80769415081219910D-08  -6.00000000000000800D-01  -5.70633894159838350D-01  -1.85410244689926460D-01
  15-TH TRUNCATION-COEFFICIENT =    1.07937460D-03
  16-TH TRUNCATION-COEFFICIENT =    6.93680565D-03
  17-TH TRUNCATION-COEFFICIENT =    2.28388239D-02
THETA =   -0.                ,  B-SUB-THETA =   -0.
```

STABLE

METHOD W(-7)

ORDER 13

```
INPUT

STEP NUMBER  K =      12
METHOD CLASS  J =       2
NUMBER OF PREDETERMINE PARAMETERS   NP =     -7
KR=-INPUT NP =     7.   NP IS RESET TO K-2 =    10
    1-TH PRELIMINARY ROOT IS     1.0000000000000000D+00   -0.
    2-TH PRELIMINARY ROOT IS     1.0000000000000000D+00   -0.
    3-TH PPELIMINARY ROOT IS     7.0000000000000000D-01    9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS     7.0000000000000000D-01    1.0800000000000000D+02
    5-TH PRELIMINARY ROOT IS     7.0000000000000000D-01    1.2600000000000000D+02
    6-TH PRELIMINARY ROOT IS     7.0000000000000000D-01    1.4400000000000000D+02
    7-TH PRELIMINARY ROOT IS     7.0000000000000000D-01    1.6200000000000000D+02

NUMBER OF ARBITRARY PARAMETERS   NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                         TO    -0.                    BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                         TO    -0.                    BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   -1.7196257477382941D+00   -9.6855656785147289D-01    1.0348794299864140+00
   2.4885564266525799D+00    2.3168794561166209D+00    9.1092128368820963D-01   -4.6805688983022413D-01
  -1.0508458630278880D+00   -8.6705933945534025D-01   -4.7091069113873375D-01   -1.5793397250187466D-01
  -2.8247524900000000D-02

   5.2621506036380960D-02    1.2866221511014238D+00    3.3801533367459571D+00    8.2976819824392245D+00
   6.1713878095457131D+00    1.2189439038563833D+01    5.8795476201930563D+00    6.6241117655518075D+00
   1.6372424674652942D+00    1.2962117932955695D+00    1.5549002457758930D-01    5.7898226050484721D-02
   2.5948367399967170D-04
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00    0.                         9.9999999999988599D-01    0.
  -6.6573954318647806D-01   -2.1631195213824752D-01   -5.6631186525651963D-01   -4.1144971900550487D-01
  -4.1144963950402361D-01   -5.6631192301770187D-01   -2.1631185867840860D-01   -6.6573957355316171D-01
   3.2757624051804151D-08   -7.0000000000177135D-01   -5.6631186525641272D-01    4.1144971900555615D-01
  -4.1144963950402234D-01    5.6631192301770415D-01   -2.1631185867943121D-01    6.6573957355830019D-01
   3.2756431759475626D-08    7.0000000000000093D-01   -6.6573954318647807D-01    2.1631195213824754D-01
   15-TH TRUNCATION-COEFFICIENT =    1.01494050D-03
   16-TH TRUNCATION-COEFFICIENT =    6.54042517D-03
   17-TH TRUNCATION-COEFFICIENT =    2.15832468D-02

THETA =    -0.                       ,   B-SUB-THETA =    -0.
```

STABLE

METHOD W(.8)

ORDER 13

```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -7
KR=-INPUT NP =     7.  NP IS RESET TO K-2 =   10
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    8.0000000000000000D-01   9.0000000000000000D+01
    4-TH PRELIMINARY ROOT IS    8.0000000000000000D-01   1.0800000000000000D+02
    5-TH PRELIMINARY ROOT IS    8.0000000000000000D-01   1.2600000000000000D+02
    6-TH PRELIMINARY ROOT IS    8.0000000000000000D-01   1.4400000000000000D+02
    7-TH PRELIMINARY ROOT IS    8.0000000000000000D-01   1.6200000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                    TO    -0.                    BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                    TO    -0.                    BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -2.2510008545580504D+00  -2.1735027001287137D+00   8.1673252990230818D-02
   2.9163037287205920D+00   4.1303393955168588D+00   2.8725244171283890D+00   4.3686408443618241D-01
  -1.3980668543609855D+00  -1.7919586374184807D+00  -1.2173504191949727D+00  -4.9845123073104996D-01
  -1.0737418240000000D-01

   5.1981297009314541D-02   1.3225914966968923D+00   4.0333856282495686D+00   1.0622832673731598D+01
   1.3119253303316511D+01   1.9249235649141460D+01   1.3340727105968801D+01   1.2768306728262307D+01
   5.3381456728490907D+00   3.0859578052712896D+00   6.9515056632674082D-01   1.7207916660066157D-01
   4.2729517103827095D-03
METHOD IS STABLE
```

STABLE

```
ROOTS ARE

   1.0000000000000000D+00   0.                       9.9999999999371423D-01   0.
  -7.6084519221311773D-01  -2.4721365958656891D-01  -6.4721356029316530D-01  -4.7022825029200556D-01
  -4.7022815943315984D-01  -6.4721362630594499D-01  -2.4721355277532411D-01  -7.6084522691789910D-01
   3.7437284630633264D-08  -8.0000000000202440D-01  -6.4721356029304311D-01   4.7022825029206417D-01
  -4.7022815943316839D-01   6.4721362630594759D-01  -2.4721355277649282D-01   7.6084522692377165D-01
   3.7435922010829273D-08   8.0000000000000106D-01  -7.6084519221311780D-01   2.4721365958656862D-01
   15-TH TRUNCATION-COEFFICIENT =    9.04365259D-04
   16-TH TRUNCATION-COEFFICIENT =    5.88995935D-03
   17-TH TRUNCATION-COEFFICIENT =    1.96054298D-02

THETA =    -0.                   ,  B-SUB-THETA =    -0.
```

```
INPUT                                                                      METHOD RW(.1)

STEP NUMBER  K =     12                                                    ORDER 13
METHOD CLASS  J =     2
NUMBER OF PREDETERMINE PARAMETERS  NP =    -7
KR=-INPUT NP =    7.  NP IS RESET TO K-2 =   10
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    1.0000000000000000D-01   1.4000000000000000D+02
    4-TH PRELIMINARY ROOT IS    1.0000000000000000D-01   1.4800000000000000D+02
    5-TH PRELIMINARY ROOT IS    1.0000000000000000D-01   1.5600000000000000D+02
    6-TH PRELIMINARY ROOT IS    1.0000000000000000D-01   1.6400000000000000D+02
    7-TH PRELIMINARY ROOT IS    1.0000000000000000D-01   1.7200000000000000D+02
NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                           TO    -0.                      BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                           TO    -0.                      BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE                                                                                      2-40

  -1.0000000000000000D+00   1.1041664789822098D+00   4.2131278515582882D-01  -2.4811812281255398D-01
  -2.0006700513293408D-01  -6.3456631830866688D-02  -1.2133019155268793D-02  -1.5567947243533907D-03
  -1.3876152476318846D-04  -8.5675613768773564D-06  -3.5253758645939570D-07  -8.7583352101779016D-09
  -1.0000000000000000D-10

   5.6205876750428289D-02   1.0872747715197362D+00   2.6524945866571655D-01   1.7320328208718515D+00
  -2.1367897489836062D+00   2.8804607144360653D+00  -2.7649934778127888D+00   2.0261221655862278D+00
  -1.1045315909065949D+00   4.3528595472295897D-01  -1.1728522703197841D-01   1.9352428413134914D-02
  -1.4761298998295656D-03
METHOD IS STABLE                                                                                      STABLE

ROOTS ARE

   1.0000064860593794D+00   0.                                9.9999351355531186D-01   0.
  -1.7789055809801692D-01   0.                               -1.7514124180436826D-02   6.6301874734980720D-02
  -1.1751621111291270D-01   1.0046230919378151D-01  -1.1751621111291270D-01  -1.0046230919378151D-01
  -6.2555715014888566D-02  -1.0401436222902941D-01  -1.6138543095904147D-01  -6.0709230698376012D-02
  -1.6138543095859916D-01   6.0709230698069839D-02  -6.2555715014888696D-02   1.0401436222902937D-01
  -1.7514124180728569D-02  -6.6301874735279424D-02
   15-TH TRUNCATION-COEFFICIENT =    1.33839063D-03
   16-TH TRUNCATION-COEFFICIENT =    8.58185553D-03
   17-TH TRUNCATION-COEFFICIENT =    2.81909447D-02

THETA =    -0.                       ,  B-SUB-THETA =    -0.
RUN COMPLETED
```

INPUT   METHOD RW(.2)

ORDER 13

STEP NUMBER K = 12
METHOD CLASS J = 2
NUMBER OF PREDETERMINE PARAMETERS NP = -7
KR=-INPUT NP = 7. NP IS RESET TO K-2 = 10

1-TH PRELIMINARY ROOT IS	1.0000000000000000D+00	-0.
2-TH PRELIMINARY ROOT IS	1.0000000000000000D+00	-0.
3-TH PRELIMINARY ROOT IS	2.0000000000000000D-01	1.4000000000000000D+02
4-TH PRELIMINARY ROOT IS	2.0000000000000000D-01	1.4800000000000000D+02
5-TH PRELIMINARY ROOT IS	2.0000000000000000D-01	1.5600000000000000D+02
6-TH PRELIMINARY ROOT IS	2.0000000000000000D-01	1.6400000000000000D+02
7-TH PRELIMINARY ROOT IS	2.0000000000000000D-01	1.7200000000000000D+02

NUMBER OF ARBITRARY PARAMETERS NA = -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM -0. TO -0. BY -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM -0. TO -0. BY -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

-1.0000000000000000D+00	2.0833295796441968D-01	1.1019170565521546D+00	4.2722208656829716D-01
-2.4471084643919626D-01	-3.0114478486024910D-01	-1.4209661657689125D-01	-4.0662816297131090D-02
-7.7615337230703586D-03	-1.0052771671997935D-03	-5.5739754505994213D-05	-4.3818676276110856D-06
-1.0240000000000000D-07			

5.4932372645104976D-02	1.1537329495174389D+00	1.1630424762975534D+00	2.6375917953480138D+00
-1.2440778152566518D+00	2.8479694854284190D+00	-2.4238717644283754D+00	1.8429851053237304D+00
-9.9733929832744961D-01	3.9433078691593199D-01	-1.0629279992298231D-01	1.7553420169860704D-02
-1.3395915053938053D-03			

METHOD IS STABLE   STABLE

ROOTS ARE

1.0000000000000000D+00	0.	9.9999999999992984D-01	0.
-1.9805361125905745D-01	-2.7834637903995641D-02	-1.9225233453453964D-01	-5.5127487591416916D-02
-1.8270908493063617D-01	-8.1347343434867835D-02	-1.6960961107565521D-01	-1.0598386589641484D-01
-1.5320887926582837D-01	-1.2855753308969826D-01	-1.9805361125908024D-01	2.7834637904029748D-02
-1.9225233448674475D-01	5.5127487556963036D-02	-1.8270908489805018D-01	8.1347343540886222D-02
-1.6960961107565682D-01	1.0598386589837356D-01	-1.5320887926582817D-01	1.2855753308969635D-01

15-TH TRUNCATION-COEFFICIENT = 1.22243358D-03
16-TH TRUNCATION-COEFFICIENT = 7.84464371D-03
17-TH TRUNCATION-COEFFICIENT = 2.57910464D-02

THETA = -0. , B-SUB-THETA = -0.

METHOD RW(.3)

ORDER 13


```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -7
KR=-INPUT NP =     7.  NP IS RESET TO K-2 =   10
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.4000000000000000D+02
    4-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.4800000000000000D+02
    5-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.5600000000000000D+02
    6-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.6400000000000000D+02
    7-TH PRELIMINARY ROOT IS    3.0000000000000000D-01   1.7200000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                        TO   -0.                 BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                        TO   -0.                 BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -6.8750056305337048D-01   1.0418128141889775D+00   1.4680619408169525D+00
   4.1682259971363170D-01  -4.1991788837502443D-01  -4.8138603783023191D-01  -2.4378490028574187D-01
  -7.6200265082862896D-02  -1.5654132724883985D-02  -2.0831453555041863D-03  -1.6451711194193164D-04
  -5.9049000000000000D-06

   5.3766051455789084D-02   1.2177747070417332D+00   2.1234767337989817D+00   4.2717834162676353D+00
   6.6334870818554809D-01   3.6379246572944315D+00  -1.7705066372993622D+00   1.7975514544853500D+00
  -8.8314584913290917D-01   3.6268674237208274D-01  -9.6614171358931628D-02   1.6030980069052179D-02
  -1.2227649262445100D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                        9.9999999998284383D-01   0.
  -2.7406362751379912D-01  -1.2202101518189546D-01  -2.8837850173035828D-01  -8.2691231335336755D-02
  -2.9708041688848881D-01  -4.1751956855726303D-02  -2.9708041668855682D-01   4.1751956855898630D-02
  -2.8837850173035748D-01   8.2691231335336909D-02  -2.7406362739526667D-01   1.2202101515245653D-01
  -2.5441441662585549D-01   1.5897579885061905D-01  -2.2981331889874225D-01   1.9283629963454754D-01
  -2.5441441661131190D-01  -1.5897579885255266D-01  -2.2981331889874225D-01  -1.9283629963454753D-01
   15-TH TRUNCATION-COEFFICIENT =    1.12226817D-03
   16-TH TRUNCATION-COEFFICIENT =    7.20706761D-03
   17-TH TRUNCATION-COEFFICIENT =    2.37129115D-02

THETA =   -0.                    ,  B-SUB-THETA =   -0.
```

STABLE

```
INPUT                                                                                                    METHOD RW(.4)

STEP NUMBER  K =     12                                                                                  ORDER 10
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -7
KR=-INPUT NP =     7.  NP IS RESET TO K-2 =    10
    1-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    2-TH PRELIMINARY ROOT IS    1.0000000000000000D+00  -0.
    3-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.4000000000000000D+02
    4-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.4800000000000000D+02
    5-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.5600000000000000D+02
    6-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.6400000000000000D+02
    7-TH PRELIMINARY ROOT IS    4.0000000000000000D-01   1.7200000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                      BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                      TO   -0.                      BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

 -1.0000000000000000D+00  -1.5833340840711606D+00   2.4100005806629729D-01   2.3164427406078112D+00
  1.9585119642707792D+00   1.8016763769255896D-01  -8.4966811118911882D-01  -7.7244993124538210D-01
 -3.6258189018654575D-01  -1.0616522119170484D-01  -1.9679646528197643D-02  -2.1386586253368758D-03
 -1.0485760000000000D-04

  5.2692769347830188D-02   1.2796618375380912D+00   3.1436804902303743D+00   6.6852804327454369D+00
  4.1337073095775022D+00   6.1289358890093744D+00  -8.4842183206357443D-02   2.2293031252563046D+00
 -6.5074530761787662D-01   3.6140959779117085D-01  -8.5132482938556844D-02   1.4932216077114526D-02
 -1.1161043736203912D-03
METHOD IS STABLE                                                                                              STABLE

ROOTS ARE

  1.0000000000000000D+00   0.                        9.9999999999943034D-01   0.
 -3.9610722252311079D-01  -5.5669275811899038D-02  -3.8450466906907929D-01  -1.1025497518283364D-01
 -3.6541816986127234D-01  -1.6269468686973577D-01  -3.3921922215131041D-01  -2.1196773179682968D-01
 -3.0641775853165675D-01  -2.5711506617939651D-01  -3.9610722251816047D-01   5.5669275808059497D-02
 -3.8450466897348951D-01   1.1025497511392607D-01  -3.6541816979610036D-01   1.6269468708177244D-01
 -3.3921922215131364D-01   2.1196773179674712D-01  -3.0641775853165634D-01   2.5711506617939671D-01
  15-TH TRUNCATION-COEFFICIENT =    1.03508138D-03
  16-TH TRUNCATION-COEFFICIENT =    6.65152042D-03
  17-TH TRUNCATION-COEFFICIENT =    2.19001371D-02

THETA =   -0.                   ,  B-SUB-THETA =   -0.
RUN COMPLETED
```

METHOD WA(.30

ORDER 13

```
INPUT

STEP NUMBER  K =      12
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -8
KR=-INPUT NP =     8.  NP IS RESET TO K-2 =    10
     1-TH PRELIMINARY ROOT IS    0.                         -0.
     2-TH PRELIMINARY ROOT IS    0.                         -0.
     3-TH PRELIMINARY ROOT IS    1.0000000000000000D+00     -0.
     4-TH PRELIMINARY ROOT IS    1.0000000000000000D+00     -0.
     5-TH PRELIMINARY ROOT IS    3.0000000000000000D-01     9.0000000000000000D+01
     6-TH PRELIMINARY ROOT IS    3.0000000000000000D-01     1.1500000000000000D+02
     7-TH PRELIMINARY ROOT IS    3.0000000000000000D-01     1.4000000000000000D+02
     8-TH PRELIMINARY ROOT IS    3.0000000000000000D-01     1.6500000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                          TO    -0.                    BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                          TO    -0.                    BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00    7.0724698329892431D-01    6.9562056089664499D-01    7.0428624426797143D-02
  -2.0068624789373265D-01   -1.6612357492361020D-01   -7.6201380782835589D-02   -2.4019309644068477D-02
  -5.3889484289444335D-03   -8.1119694917508418D-04   -6.5610000000000000D-05   -0.
  -0.

   5.5645753656849570D-02    1.1166282631067988D+00    6.6537770462037520D-01    2.1594925703214180D+00
  -1.7091918302017214D+00    2.8916144326407512D+00   -2.6048521866609252D+00    1.9492057657655172D+00
  -1.0571025583294093D+00    4.1742837497328003D-01   -1.1247896620425538D-01    1.8565444156336681D-02
  -1.4164549748348995D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00    0.                        9.9999999999996046D-01    0.
  -2.8977774122544367D-01   -7.7645738390975489D-02   -2.2981331889874182D-01    1.9283629963454760D-01
  -2.2981331889874220D-01   -1.9283629963454745D-01   -1.2678546226469693D-01   -2.7189234369217282D-01
   1.4038274314128302D-08   -3.0000000000020248D-01   -1.2678546226482061D-01    2.7189234369193944D-01
   1.4038468985777475D-08    2.9999999999999967D-01   -2.8977774122544367D-01    7.7645738390975494D-02
   15-TH TRUNCATION-COEFFICIENT =    1.28763325D-03
   16-TH TRUNCATION-COEFFICIENT =    8.25912720D-03
   17-TH TRUNCATION-COEFFICIENT =    2.71402361D-02

THETA =    -0.                    ,  B-SUB-THETA =    -0.
```

STABLE

METHOD WA(.5)

ORDER 13

```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -8
KR=-INPUT NP =     8.  NP IS RESET TO K-2 =    10
    1-TH PRELIMINARY ROOT IS    0.                        -0.
    2-TH PRELIMINARY ROOT IS    0.                        -0.
    3-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    4-TH PRELIMINARY ROOT IS    1.0000000000000000D+00   -0.
    5-TH PRELIMINARY ROOT IS    5.0000000000000000D-01    9.0000000000000000D+01
    6-TH PRELIMINARY ROOT IS    5.0000000000000000D-01    1.1500000000000000D+02
    7-TH PRELIMINARY ROOT IS    5.0000000000000000D-01    1.4000000000000000D+02
    8-TH PRELIMINARY ROOT IS    5.0000000000000000D-01    1.6500000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                          TO    -0.                  BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                          TO    -0.                  BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00  -1.5458836116845949D-01   8.3727263204384564D-01   8.6056563477352068D-01
   2.7445224848209470D-01  -1.8891366977887819D-01  -3.0111894383439880D-01  -2.0684097801766464D-01
  -9.1069369356802725D-02  -2.5652943143257180D-02  -3.3062500000000000D-03  -0.
  -0.

   5.4491993714609680D-02   1.1789427802523036D+00   1.5719166637502215D+00   3.5456450843622353D+00
  -3.6036209330938565D-02   3.6138148094630561D+00  -1.9054016861107785D+00   1.9426714928050433D+00
  -9.2632639838519450D-01   3.8763285476547125D-01  -1.0272572172328507D-01   1.7000172447122237D-02
  -1.2979244433861294D-03
METHOD IS STABLE

ROOTS ARE

   1.0000000000000000D+00   0.                       9.9999999997949063D-01   0.
  -4.8296290204240601D-01  -1.2940956398495909D-01  -3.8302219816456970D-01   3.2139383272424599D-01
  -3.8302219816457033D-01  -3.2139383272424576D-01  -2.1130910377449488D-01  -4.5315390615362137D-01
   2.3397123856880047D-08  -5.0000000000033746D-01  -2.1130910377470102D-01   4.5315390615323240D-01
   2.3397448309628881D-08   4.9999999999999945D-01  -4.8296290204240611D-01   1.2940956398495916D-01
   15-TH TRUNCATION-COEFFICIENT =    1.18644223D-03
   16-TH TRUNCATION-COEFFICIENT =    7.61532967D-03
   17-TH TRUNCATION-COEFFICIENT =    2.50428525D-02

THETA =    -0.                  ,  B-SUB-THETA =    -0.
```

STABLE

METHOD WB(.3)

ORDER 13

```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =     2
NUMBER OF PREDETERMINE PARAMETERS  NP =    -9
KR=-INPUT NP =    9.  NP IS RESET TO K-2 =   10
    1-TH PRELIMINARY ROOT IS    0.                        -0.
    2-TH PRELIMINARY ROOT IS    0.                        -0.
    3-TH PRELIMINARY ROOT IS    0.                        -0.
    4-TH PRELIMINARY ROOT IS    0.                        -0.
    5-TH PRELIMINARY ROOT IS    1.0000000000000000D+00    -0.
    6-TH PRELIMINARY ROOT IS    1.0000000000000000D+00    -0.
    7-TH PRELIMINARY ROOT IS    3.0000000000000000D-01    9.0000000000000000D+01
    8-TH PRELIMINARY ROOT IS    3.0000000000000000D-01    1.2000000000000000D+02
    9-TH PRELIMINARY ROOT IS    3.0000000000000000D-01    1.5000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =    -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                    TO   -0.                    BY   -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM   -0.                    TO   -0.                    BY   -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   1.1803848416241806D+00   2.1334579094812179D-01  -1.1537683089967220D-01
  -1.6915268486406003D-01  -7.7510392269097768D-02  -2.5780841756628249D-02  -5.1808827828441367D-03
  -7.2900000000000000D-04  -0.                      -0.                      -0.
  -0.

   5.6339508284475723D-02   1.0810659856355237D+00   2.0224388824983561D-01   1.7914180225748648D+00
  -2.1132879371107309D+00   2.9403387054399745D+00  -2.7928473178638636D+00   2.0511382827825051D+00
  -1.1174489308950008D+00   4.4031405119499360D-01  -1.1862509200544230D-01   1.9571435996200394D-02
  -1.4927038679521293D-03
METHOD IS STABLE
```

STABLE

```
ROOTS ARE

   1.0000000000000000D+00   0.                       9.9999999999922286D-01   0.
  -2.5980760943560629D-01  -1.5000020262784420D-01  -1.4999998379199417D-01  -2.5980763049344242D-01
   1.4038468859332846D-08  -2.9999999999999957D-01  -1.4999998379003552D-01   2.5980763049449746D-01
   1.4037896283314585D-08   3.0000000000013672D-01  -2.5980760943660665D-01   1.5000020262784160D-01
   15-TH TRUNCATION-COEFFICIENT =    1.35218856D-03
   16-TH TRUNCATION-COEFFICIENT =    8.66935214D-03
   17-TH TRUNCATION-COEFFICIENT =    2.84750146D-02

THETA =    -0.                 ,  B-SUB-THETA =    -0.
```

METHOD WB(.5)

ORDER 13

```
INPUT

STEP NUMBER  K =     12
METHOD CLASS  J =      2
NUMBER OF PREDETERMINE PARAMETERS  NP =     -9
KR=-INPUT NP =     9.  NP IS RESET TO K-2 =    10
    1-TH PRELIMINARY ROOT IS     0.                          -0.
    2-TH PRELIMINARY ROOT IS     0.                          -0.
    3-TH PRELIMINARY ROOT IS     0.                          -0.
    4-TH PRELIMINARY ROOT IS     0.                          -0.
    5-TH PRELIMINARY ROOT IS     1.0000000000000000D+00      -0.
    6-TH PRELIMINARY ROOT IS     1.0000000000000000D+00      -0.
    7-TH PRELIMINARY ROOT IS     5.0000000000000000D-01   9.0000000000000000D+01
    8-TH PRELIMINARY ROOT IS     5.0000000000000000D-01   1.2000000000000000D+02
    9-TH PRELIMINARY ROOT IS     5.0000000000000000D-01   1.5000000000000000D+02

NUMBER OF ARBITRARY PARAMETERS  NA =     -0
THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                      TO    -0.                   BY    -0.
B-SUB-THETA IS AN ARBITRARY PARAMETER WHOSE VALUES STEP
FROM    -0.                      TO    -0.                   BY    -0.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

COEFFICIENTS ARE

  -1.0000000000000000D+00   6.3397473604030108D-01   5.4903795624296197D-01   3.1698726767610487D-01
  -1.1274049116140895D-01  -1.7688290487633119D-01  -1.4062498492414660D-01  -5.4126578997481182D-02
  -1.5625000000000000D-02  -0.                      -0.                      -0.
  -0.

   5.5573771342002195D-02   1.1213513076130162D+00   7.5647277360231479D-01   2.4235979842469456D+00
  -1.4415869422472151D+00   3.0348225346422902D+00  -2.5419347345905906D+00   1.9532511271250119D+00
  -1.0535332807963722D+00   4.1578300293262159D-01  -1.1208045626184027D-01   1.8499513066720614D-02
  -1.4114314031119918D-03
METHOD IS STABLE
```

STABLE

```
ROOTS ARE

   1.0000000000000000D+00   0.                       9.9999999999989940D-01   0.
  -2.4999997296743483D-01  -4.3301271747556627D-01  -4.3301268239165155D-01  -2.5000003377083012D-01
  -2.4999997298295323D-01   4.3301271749051678D-01  -4.3301268238801933D-01   2.5000003377094133D-01
   2.3397448752133370D-08   4.9999999999999897D-01   2.3397448309616525D-08  -4.9999999999999945D-01
   15-TH TRUNCATION-COFFFICIENT =    1.28305597D-03
   16-TH TRUNCATION-COEFFICIENT =    8.22976657D-03
   17-TH TRUNCATION-COEFFICIENT =    2.70437705D-02

THETA =    -0.                    ,   B-SUB-THETA =    -0.
```

2.2 Comparison of Methods on Circular Trajectory

Traditional Two Body Integrator

All tests were made on a circular orbit, with radius of 8×10^6 meters and period slightly less than two hours. Störmer predictors of the same order as the corrector were used in each case. Length of the position error vector was recorded after 84 revolutions (approximately one week). Results are shown in the following table:

TABLE 2-1

COMPARISONS OF PROPAGATED ERROR

Method - Order	Algorithm	Step-size H (seconds)	Maximum Absolute Position Error (meters)
Adams - order 13	PEC	40	.00328
" " "	"	60	.00207
" " "	"	80	Unstable
" " "	PECE	180	.14961
" " "	"	240	4.47145
" " "	"	300	39.75582
Adams - order 12	PEC	60	.00224
" " "	"	80	.00167
" " "	"	100	Unstable
" " "	PECE	180	.88490
" " "	"	240	35.81414
" " "	"	300	604.96493
Adams - order 11	PEC	100	.01548
" " "	"	120	.12669
" " "	PECE	180	8.32973
" " "	"	240	164.60818
" " "	"	300	1399.27667
Adams - order 10	PEC	100	.06671
" " "	"	120	.52421
" " "	PECE	180	34.99581
" " "	"	240	823.07395
" " "	"	300	9329.39614

TABLE 2-1 (cont'd)

Method - Order	Algorithm	Step-size H (seconds)	Maximum Absolute Position Error (meters)
Adams - order 9	PEC	100	2.78891
" " "	"	120	14.83160
" " "	PECE	180	460.42099
" " "	"	240	4670.17085
" " "	"	300	36581.38435
Adams - order 8	PEC	100	7.53281
" " "	"	120	41.92423
Adams - order 7	PEC	100	476.78478
" " "	"	120	1766.14615
" " "	PECE	180	26082.05203
" " "	"	240	189349.40678
" " "	"	300	853255.62828
W(.7) - order 13	PEC	40	.10153
" " "	"	60	.00105
" " "	"	80	Unstable
" " "	PECE	180	Unstable
W(.7) - order 12	PEC	60	.00066
" " "	"	80	.05213
" " "	"	100	Unstable
" " "	PECE	180	Unstable
W(.7) - order 11	PEC	100	.00251
" " "	"	120	.02488
" " "	PECE	180	.29679
" " "	"	240	6.49083
" " "	"	300	64.60674
W(.7) - order 10	PEC	100	.01416
" " "	"	120	.13316
" " "	PECE	180	.49238
" " "	"	240	13.19697
" " "	"	300	148.69498
W(.7) - order 9	PEC	100	.51177
" " "	"	120	3.23379
" " "	PECE	180	40.37668
" " "	"	240	531.32660
" " "	"	300	3521.20609

TABLE 2-1 (cont'd)

Method - Order	Algorithm	Step-size H (seconds)	Maximum Absolute Position Error (meters)
W(.7) - order 8	PEC	100	1.51706
" " "	"	120	10.86264
W(.7) - order 7	PEC	100	124.40579
" " "	"	120	510.27195
" " "	PECE	180	5015.92314
" " "	"	240	37134.49313
" " "	"	300	168840.18203
W(.6) - order 11	PEC	100	.00159
" " "	"	120	.02616
W(.6) - order 10	PEC	100	.01581
" " "	"	120	.13899
W(.6) - order 9	PEC	100	.61817
" " "	"	120	3.78216
W(.8) - order 11	PEC	100	.00265
" " "	"	120	.02362
W(.8) - order 10	PEC	100	.01285
" " "	"	120	.13074
W(.8) - order 9	PEC	100	.42897
" " "	"	120	2.80941
W(.2) - order 13	PECE	180	.05462
" " "	"	240	1.72194
" " "	"	300	20.70266
W(.3) - order 13	PECE	180	.03113
" " "	"	240	1.01814
" " "	"	300	Unstable
" " "	PEC	100	Unstable
RW(.3) - order 12	PEC	60	.00580
" " "	"	80	.00426
" " "	PECE	180	.04427
" " "	"	240	Unstable

TABLE 2-1 (cont'd)

Method - Order	Algorithm	Step-size H (seconds)	Maximum Absolute Position Error (meters)
RW(.3) - order 11	PEC	100	.00041
" " "	"	120	.03055
" " "	PECE	180	1.05460
" " "	"	240	22.79531
" " "	"	300	221.71056
RW(.3) - order 10	PEC	100	.02060
" " "	"	120	.16125
" " "	PECE	180	2.93285
" " "	"	240	73.61465
" " "	"	300	861.82434
RW(.3) - order 9	PEC	100	.78742
" " "	"	120	4.64361
" " "	PECE	180	92.96631
" " "	"	240	1193.15883
" " "	"	300	8060.42573
RW(.3) - order 8	PEC	100	2.10032
" " "	"	120	13.89388
RW(.3) - order 7	PEC	100	187.16553
" " "	"	120	734.67982
" " "	PECE	180	8818.52277
" " "	"	240	65087.05119
" " "	"	300	297633.04892
RW(.1) - order 13	PECE	180	.06844
" " "	"	240	2.08295
" " "	"	300	23.66179
RW(.2) - order 10	PEC	100	.02674
" " "	"	120	.20109
RW(.2) - order 9	PEC	100	1.11480
" " "	"	120	6.31428
RW(.4) - order 10	PEC	100	.01548
" " "	"	120	6.22581
" " "	"	100	.59986
" " "	"	120	3.68466

TABLE 2-1 (cont'd)

Method - Order	Algorithm	Step-size H (seconds)	Maximum Absolute Position Error (meters)
WA(.3) - order 13	PECE	180	.04285
" " "	"	240	1.38030
" " "	"	300	17.24420
WA(.3) - order 12	PECE	180	.18970
" " "	"	240	8.11058
" " "	"	300	140.80248
WB(.3) - order 13	PECE	180	.06728
" " "	"	240	2.10079
" " "	"	300	24.50652

3. Minimization of Truncation Error

The purpose of this research is to reduce the propagated truncation error in the integration of perturbed satellite orbits by balancing the local truncation error against stability. The approach taken was to minimize the constants in the local truncation error expansion, C_i, subject to conditions on the extraneous roots, r_i. The analysis applies to the traditional, Class II integrator

$$\sum_{i=0}^{k} a_i \, y_{n-i} + h^2 \sum_{i=0}^{k} b_i \, \ddot{y}_{n-i} = 0 \; .$$

As a first look at the problem, the coefficients a_i of 12 back-point, order 11, methods were chosen to place the extraneous roots r_i at reasonable places, as in methods W(.7) and RW(.3). Most of the coefficients b_i were used to solve the order equations parametrically in terms of b_0, b_1 and b_0 and b_1 were chosen to minimize some function, f, of the first few non-zero C_i. To decide what f should be, consider the ideal situation where the parametric order equation solutions, linear in b_0 and b_1, are algebraically substituted into the expressions for C_i. The C_i are, then, linear in b_0 and b_1. The local truncation error, T, can be expressed as the sum of the C_i, times the higher derivatives of the differential equation, times the powers of h. T is, therefore, linear in b_0 and b_1.

Considering constant step-size h, at a fixed place in the orbit, in a particular coordinate direction, T is a plane in R^3 which is not parallel to the b_0-b_1 plane, since T can change with b_0 and b_1.

Therefore, there is a line in the b_0-b_1 plane where $T = 0$, i.e., there is absolutely no truncation error. However, since the errors in all three directions are important, we must add their absolute values, making T nonlinear in b_0 and b_1. Also, the higher derivatives change in magnitude and sign through the orbit, so we must at least take the sum of the absolute values of the terms making T nonlinear. For the latter reason we must also assume all the higher derivatives equal if we wish to derive a method good for the whole orbit. Three expressions for f were used:

$$f = \sum_i W_i |C_i|$$

$$f = \sum_i W_i C_i^2$$

$$f = \sqrt{\sum_i W_i C_i^2} \quad .$$

The weights, W_i, were chosen to be proportional to powers of h. For example, $W_{13} = 100^2$, $W_{14} = 100$, $W_{15} = 1$ for $h = 100$ seconds.

Since these are complicated nonlinear functions, no explicit expressions were derived. The next section will describe the optimization algorithm. The following table summarizes the results of optimization. The integration error are given in another section of this report. The notation used is that of Henrici.

The C_i starting values were those obtained with the original order 13 W(.7) coefficients. It appears that more experimentation with $f = {}_i\Sigma\, W_i |C_i|$ will yield better results than obtained so far.

Since part of the reason for the successful optimization of the cyclic methods is due to allowing the r_i to vary inside the unit circle, it was felt that a similar approach might yield more significant results for traditional methods [see the midyear report for NASA - Cal. State Fullerton Grant, Dr. S. Pierce]. An effort was begun in this direction.

In addition to these computational considerations, there are analytical formulations of the problem which may yield improved methods. For example, a detailed study of the higher derivatives (for circles, ellipses, ...) could lead to improved weights and an expression for the propagated truncation error (for circles, ellipses, ...) could be optimized. The latter approach would involve the C_i, the r_i, and products of the coefficients of the algorithm arising from the propagation process and would probably take the form of a constrained optimization problem.

3.1 Algorithm Description

The basic optimization algorithm is the conjugate gradient method [Polak, Computing Methods in Optimization, Academic Press, N. Y. (1971)] with fitting of quadratic polynomials to search for the minimum. The algorithm allows for functions which must be computed numerically and whose

derivatives must be computed numerically. The algorithm was specialized to optimize traditional methods by modifying the input, output and function subroutine.

The main program reads in the number of backpoints ($K = 12$), the number of nonzero C_i to be minimized ($NC = 2,3,4$ or 5), the number of parameters used for optimization ($N = 2$), the weights (W_i), and the "a" coefficients chosen for normalization and positioning of roots W(.7) or RW(.3). Values and derivatives of the object function, f, are computed which determine the direction and length of the optimizing step to be taken in the parameter space.

The function subroutine uses the present values of b_0 and b_1 and the given a_i to solve the order equations $C_2 = \cdots = C_{12} = 0$ for $b_2,\ldots,b_{12}$. The a_i and computed b_i are substituted into the expressions for $C_{13},C_{14},\ldots,C_{12+NC}$. These are combined with the W_i to compute f.

We worked with 12 step, order 11 methods using two parameters to minimize f. At each iteration, the program prints the function value, the C_i, and the present values of b_0 and b_1. At the end of the run the final values of all the a_i, b_i and C_i are printed.

Two methods resulting from this analysis have been tested in the integration of the usual circular trajectory with two-hour period (one week arc). These methods are optimized versions of W(.7), order 13. They are the second and third methods appearing in Table 3-1 below. Computations were done with different step sizes and with algorithms

PEC and PECE. In all computations, optimized methods showed a consistent, but limited, error reduction when compared with the basic $W(.7)$ method. For example, the position errors (meters) in computations with the third method of Table 3-1 were compared with $W(.7)$ (algorithm PECE). The error reduction at $h = 100$ secs. was seven percent and at $h = 140$ secs. fifteen percent. The order 13 Störmer predictor was used throughout. Little change in stability regions was observable.

Interesting, although basic, research problems suggest themselves as a result of this experiment. It is believed that significant improvement in efficiency can probably be obtained by deriving several methods, depending on orbit type, orbit plane orientation and vehicle position, and by considering variable method algorithms.

It is to be noted that the first three non-vanishing error coefficients of method $W(.7)$, order 11, are:

$$C_{13} = 1.25 \times 10^{-3}$$
$$C_{14} = 6.83 \times 10^{-3}$$
$$C_{15} = 1.91 \times 10^{-2} \quad .$$

These coefficients can be compared with those of the (11^{th} order) methods given in Table 3-1.

TABLE 3-1

12 BACKPOINT, ORDER 11, OPTIMIZED TRADITIONAL METHODS

Coefficients	f	W_{13}	W_{14}	W_{15}	W_{16}	W_{17}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	Final f Value
W(.7)	$\Sigma W_i c_i^2$	1	1	0	0	0	2.6E-5	4.1E-6				7.2E-10
W(.7)	$\Sigma W_i c_i^2$	100^2	100	1	0	0	1.5E-6	9.4E-6	9.8E-4			1.0E-7
W(.7)	$\Sigma W_i c_i^2$	100^3	100^2	100	1	0	1.7E-6	1.1E-5	9.8E-4	6.5E-3		1.4E-4
W(.7)	$\Sigma W_i c_i^2$	10^4	10^3	10^2	10	1	5.6E-5	3.6E-4	1.8E-4	3.9E-3	1.7E-2	6.0E-4
RW(.3)	$\Sigma W_i c_i^2$	5^4	5^3	5^2	5	1	1.0E-3	3.1E-3	2.0E-3	8.2E-3	2.8E-2	3.1E-3
W(.7)	$\sqrt{\Sigma W_i c_i^2}$	1	1	0	0	0	2.6E-5	4.1E-6				2.7E-5
W(.7)	$\Sigma W_i \lvert c_i \rvert$	1	0	0	0	0	3.2E-10					3.2E-10
W(.7)	$\Sigma W_i \lvert c_i \rvert$	10^2	10	1	0	0	7.0E-12	1.7E-4	8.5E-6			1.7E-3
W(.7)	$\Sigma W_i \lvert c_i \rvert$	1	1	0	0	0	1.4E-11	1.4E-10				1.6E-10
RW(.3)	$\Sigma W_i c_i^2$	1	1	0	0	0	5.5E-4	8.6E-5				3.1E-7
RW(.3)	$\Sigma W_i c_i^2$	10^2	10	1	0	0	7.0E-4	8.5E-4	5.9E-3			
RW(.3)	$\Sigma W_i c_i^2$	100^2	100	1	0	0	1.9E-4	2.4E-3	1.7E-2			1.2E-2
W(.7)	Starting	Values					9.8E-11	6.2E-10	1.0E-3	6.5E-3	2.2=-2	

4. Traditional Methods in Summed Coordinate Form

This section describes the general mathematical procedure and programming for converting the coefficients of the coordinate form of the difference equation to the summed coordinate form.

4.1 Mathematical Procedure

The coordinate form of the Class II difference equation is defined as

$$\sum_{i=0}^{k} a_i \, y_{n-i} + h^2 \sum_{i=0}^{k} b_i \, \ddot{y}_{n-i} = 0 \; .$$

The difference form of a corrector difference equation is defined as

$$\left(\sum_{i=0}^{k} \alpha_i \nabla^i \right) y_n + h^2 \left(\sum_{i=0}^{k} \beta_i \nabla^i \right) \ddot{y}_n = 0 \; .$$

Given coefficients a_i, b_i, $i = 0,1,\ldots,k$, of the coordinate form of the difference equation, we can find coefficients α_i, β_i, $i = 0,\ldots,k$, of the difference form.

Multiplying the difference form by ∇^{-2}, we have:

$$\left(\alpha_0 \nabla^{-2} + \alpha_1 \nabla^{-1} \right) y_n + \left(\sum_{i=2}^{k} \alpha_i \nabla^{i-2} \right) y_n$$

$$+ h^2 \left(\beta_0 \nabla^{-2} + \beta_1 \nabla^{-1} \right) \ddot{y}_n + h^2 \left(\sum_{i=2}^{k} \beta_i \nabla^{i-2} \right) \ddot{y}_n = 0 .$$

It turns out that $\alpha_0 = \alpha_1 = 0$. Since the functions $y \equiv 1$, $y \equiv x$, both satisfy the coordinates form of the difference equation, it follows that they also satisfy the difference form. For $y \equiv 1$,

$$\nabla^i(y) \equiv 0, \quad i > 0 \quad \text{and} \quad \ddot{y} \equiv 0 .$$

Therefore, we have

$$\alpha_0 \nabla^0 y_n \equiv 0 \quad \Rightarrow \quad \alpha_0 = 0 .$$

A similar simple argument shows that $\alpha_1 = 0$ also.

We have:

$$\left(\sum_{i=2}^{k} \alpha_i \nabla^{i-2} \right) y_n + h^2 \left(\beta_0 \nabla^{-2} + \beta_1 \nabla^{-1} \right) \ddot{y}_n \qquad (4\text{-}1)$$

$$+ h^2 \left(\sum_{i=2}^{k} \beta_i \nabla^{i-2} \right) y_n = 0 .$$

Reconverting Equation (4-1) partially, to the coordinate form, we get:

$$\sum_{i=0}^{k-2} \bar{\alpha}_i \, y_{n-i} + h^2 \sum_{i=0}^{k-2} \bar{\beta}_i \, \ddot{y}_{n-i} \tag{4-2}$$

$$+ h^2 \left(\beta_0 \, \nabla^{-2} + \beta_1 \, \nabla^{-1} \right) \ddot{y}_n = 0 \; .$$

Equation (4-2) is called the summed form.

The Predictor

The summed form of the predictor equation is:

$$\sum_{i=0}^{k-2} \bar{\alpha}_i \, y_{n-i} + h^2 \sum_{i=1}^{k-2} \bar{\beta}_{i-1} \, \ddot{y}_{n-i} + h^2 (\beta_0 \, \nabla^{-2} + \beta_1 \, \nabla^{-1}) \ddot{y}_{n-1} = 0 \; .$$

The coefficients $\bar{\alpha}_i$, $\bar{\beta}_i$, $i = 0,1,\ldots,k-2$, are obtained from converting coefficients a_i, b_i, $i = 0,1,\ldots,k$, of the coordinate form by the following steps:

1) $a_i' = a_{i+1}$, $b_i' = b_{i+1}$, $i = 0,1,\ldots,k$.

2) α_i, β_i, $i = 0,\ldots,k$ are computed as:

$$C = A \, Y^T$$

$$D = B \, Y^T$$

where

$$C = \{\alpha_0,\alpha_1,\ldots,\alpha_k\}, \qquad D = \{\beta_0,\beta_1,\ldots,\beta_k\},$$

$$A = \{a'_0,a'_1,\ldots,a'_k\}, \qquad B = \{b'_0,b'_1,\ldots,b'_k\},$$

and Y is a matrix, the Pascal Triangle of the proper dimension. (The program now permits conversion of a method with $k \leq 16$.) The entries of the Pascal Triangle are obtained from the equation $a_{i,j} = a_{i-1,j} - a_{i-1,j-1}$, i being the row index and j the column index. The entries of the first row and column of the matrix are:

$$a_{i,1} = 1 \quad \text{for all} \quad i$$

$$a_{1,j} = 0 \quad \text{for} \quad j \geq 2 \ .$$

3) $\alpha'_i = \alpha_{i+2}$, $\beta'_i = \beta_{i+2}$, $\quad i = 0,1,\ldots,k$.

4) $\bar{\alpha}_i$, $\bar{\beta}_i$ are computed

$$E = C'Y^T$$

$$F = D'Y^T$$

where

$$E = \{\bar{\alpha}_0,\bar{\alpha}_1,\ldots,\bar{\alpha}_{k-2}\}, \qquad F = \{\bar{\beta}_0,\bar{\beta}_1,\ldots,\bar{\beta}_{k-2}\},$$

$$C' = \{\alpha'_0,\alpha'_1,\ldots,\alpha'_{k-2}\}, \qquad D' = \{\beta'_0,\beta'_1,\ldots,\beta'_{k-2}\}.$$

<u>The Corrector</u>

The summed form of the corrector equation is:

$$\sum_{i=0}^{k-2} \bar{\alpha}_i \, y_{n-i} + h^2 \sum_{i=0}^{k-2} \bar{\beta}_i \, \ddot{y}_{n-i} + h^2(\beta_0 \nabla^{-2} + \beta_1 \nabla^{-1})\ddot{y}_n = 0 \; .$$

The coefficients $\bar{\alpha}_i$, $\bar{\beta}_i$, $i = 0,1,\ldots,k-2$, are obtained by converting coefficients a_i, b_i, $i = 0,1,\ldots,k$, of the coordinate form by the following steps:

1) α_i, β_i, $i = 0,1,\ldots,k$ are computed

$$C = A\,Y^T$$

$$D = B\,Y^T$$

where

$$C = \{\alpha_0,\alpha_1,\ldots,\alpha_k\}, \qquad D = \{\beta_0,\beta_1,\ldots,\beta_k\},$$

$$A = \{a_0,a_1,\ldots,a_k\}, \qquad B = \{b_0,b_1,\ldots,b_k\},$$

and Y is the Pascal Triangle matrix.

2) $\alpha'_i = \alpha_{i+2}$, $\beta'_i = \beta_{i+2}$, $i = 0,1,\ldots,k$.

3) $\bar{\alpha}_i$, $\bar{\beta}_i$ are computed

$$E = C'\,Y^T$$

$$F = D'\,Y^T$$

where

$$E = \{\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{k-2}\}, \quad F = \{\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_{k-2}\},$$

$$C' = \{\alpha'_0, \alpha'_1, \ldots, \alpha'_{k-2}\}, \quad D' = \{\beta'_0, \beta'_1, \ldots, \beta'_{k-2}\}.$$

<u>The Backward Difference Operator</u>

The backward difference operators ∇^j, $j \geq 0$, of Equation (4-1) are defined in the usual manner.

The sums ∇^{-1} and ∇^{-2} are defined by

$$\nabla^{-1} \nabla = I$$

$$\nabla^{-2} \nabla^2 = I .$$

Thus, $\nabla^{-1} y_n$ and $\nabla^{-2} y_n$ are uniquely defined, up to an additive constant and linear function, respectively.

The initial $\nabla^{-1} y_n$ and $\nabla^{-2} y_n$ of Equation (4-2) are computed by the following method:

Given the following data:

$$\begin{array}{ccccc} y_n & y_{n-1} & y_{n-2} & \cdots & y_{n-k} \\ \ddot{y}_n & \ddot{y}_{n-1} & \ddot{y}_{n-2} & \cdots & \ddot{y}_{n-k} \end{array}$$

The initial sums $\nabla^{-1} y_n$ and $\nabla^{-2} y_n$ are computed by using the corrector

equation with two different data sets:

$$\sum_{i=0}^{k-2} \bar{\alpha}_i \, y_{(n-1)-i} + h^2 \sum_{i=0}^{k-2} \bar{\beta}_i \, \ddot{y}_{(n-1)-i} + h^2(\beta_0 \nabla^{-2} + \beta_1 \nabla^{-1})\ddot{y}_{n-1} = 0 \,,$$

$$\sum_{i=0}^{k-2} \bar{\alpha}_i \, y_{n-i} + h^2 \sum_{i=0}^{k-2} \bar{\beta}_i \, \ddot{y}_{n-i} + h^2(\beta_0 \nabla^{-2} + \beta_1 \nabla^{-1})\ddot{y}_n = 0 \,. \tag{4-3}$$

From (4-3) we have:

$$(\beta_0 \nabla^{-2} + \beta_1 \nabla^{-1})\ddot{y}_{n-1} = \frac{\sum_{i=0}^{k-2} \bar{\alpha}_i \, y_{(n-1)-i} + h^2 \sum_{i=0}^{k-2} \bar{\beta}_i \, \ddot{y}_{(n-1)-i}}{h^2} = -A \,,$$

$$(\beta_0 \nabla^{-2} + \beta_1 \nabla^{-1})y_n = \frac{\sum_{i=0}^{k-2} \bar{\alpha}_i \, y_{n-i} + h^2 \sum_{i=0}^{k-2} \bar{\beta}_i \, \ddot{y}_{n-i}}{h^2} = -B \,. \tag{4-4}$$

Applying the recursion formulas (which are derived from the definitions of ∇^{i} and ∇^{-i}):

$$\nabla^{-1} \ddot{y}_n = \ddot{y}_n + \nabla^{-1} \ddot{y}_{n-1} \,,$$

$$\nabla^{-2} \ddot{y}_n = \nabla^{-1} \ddot{y}_n + \nabla^{-2} \ddot{y}_{n-1} \,,$$

to (4-4) we have

$$\beta_0 \nabla^{-2} \ddot{y}_{n-1} + \beta_1 \nabla^{-1} \ddot{y}_{n-1} = - A \,,$$

$$\beta_0 \nabla^{-2} \ddot{y}_{n-1} + (\beta_0 + \beta_1)\nabla^{-1} \ddot{y}_{n-1} + (\beta_0 + \beta_1)\ddot{y}_n = - B \,.$$

Subtracting these two equations

$$- \beta_0 \nabla^{-1} \ddot{y}_{n-1} - (\beta_0 - \beta_1)\ddot{y}_n = B - A \,.$$

Finally, solving for $\nabla^{-1} \ddot{y}_{n-1}$ and $\nabla^{-2} \ddot{y}_{n-1}$, we have

$$\nabla^{-1} \ddot{y}_{n-1} = \frac{- (\beta_0 + \beta_1)\ddot{y}_n + A - B}{\beta_0} \,,$$

$$\nabla^{-2} \ddot{y}_{n-1} = \frac{- A - \beta_1 \nabla^{-1} \ddot{y}_{n-1}}{\beta_0} \,.$$

The sums $\nabla^{-1} \ddot{y}_n$ and $\nabla^{-2} \ddot{y}_n$ are computed from the initial $\nabla^{-1} \ddot{y}_{n-1}$ and $\nabla^{-2} \ddot{y}_{n-1}$ using the recursion formulas. The recursion formulas are used, also, to update the operators $\nabla^{-1} \ddot{y}_n$ and $\nabla^{-2} \ddot{y}_n$ for every $\ddot{y}_n$ computed.

4.2 Program GENRTE

Program GENRTE converts coefficients a_i, b_i, $i = 0,1,\ldots,k$ of

the corrdinate form of the difference equation into coefficients $\bar{\alpha}_i$, $\bar{\beta}_i$, $i = 0,\ldots,k$ of the summed form of the difference equation. The program consists of the main program and a subroutine CONVRT.

Input coefficients are determined as predictors or correctors and are converted by calling subroutine CONVRT, which performs the operation $AY^T = A'$

$$A = \{a_0, a_1, \ldots, a_k\}, \qquad A' = \{\alpha_0, \alpha_1, \ldots, \alpha_k\},$$

and Y being the Pascal Triangle matrix.

Program GENRTE is coded in FORTRAN double precision and coded to operate on the CDC 6000 series machines, i.e.:

Input

Item	Format	
1. K	I5	K is the step number
2. $J = 1$ or $J = 2$	I5	$J = 1$ if method is a predictor, $J = 2$ if method is a corrector.
3. a_i, $i = 0,\ldots,k$	3D25.0	
4. b_i, $i = 0,\ldots,k$	3D25.0	

Output

The printed output for each case consists of K, the step-number; J the predictor or corrector indicator; the a_i, b_i, $i = 0,1,\ldots,k$ (un-

summed coefficients, under the heading "A-B COEFFICIENTS;" the α_i, β_i, $i = 0,1,\ldots,k$ (difference form) coefficients under the heading "ALPHA-BETA COEFFICIENTS:" and the $\bar{\alpha}_i$, $\bar{\beta}_i$, $i = 0,\ldots,k$ (summed ordinate form) coefficients under the heading "ALPHA-BETA-BAR COEFFICIENTS." The punched output consists of $\bar{\alpha}_i$, $\bar{\beta}_i$, $i = 0,\ldots,k-2$, with format of (3D25.16) and β_0, β_1 with the format of (2D25.16).

Program GENRTE is also written in as part of a program, the SUMMED TWO-BODY INTEGRATOR, which has not been documented. In this program, the coefficients of the coordinate form of the difference equation are converted to the summed coordinate form and used directly in the program. The SUMMED TWO-BODY INTEGRATOR is designed to be run on the IBM 360.

Coefficients in the summed coordinate form have been generated for several methods. The coefficient generator GENRTE accepts card output from program A-B. The program was validated by comparing coefficients generated by GENRTE with those given in the GSFC document [11] for various Adams Class II methods. GENRTE output is given below.

The methods were tested against the unsummed methods on the 2-hr. circular orbit with algorithm PEC, step-sizes $h = 60$ secs., and 100 secs., and with algorithm PECE, $h = 180$ secs. and 300 secs. The arc lengths varied from 1 rev. to 84 revs. In all cases, the position error in the summed method, $e(s)$, was smaller than in the unsummed method $e(u)$, and

$$.48 < \frac{e(s)}{e(u)} < .84 .$$

The summed Störmer predictor was used with summed methods.

METHOD: ADAMS

J = 1

```
ORDER   =      9

A-B COEFFICIENTS ARE

 -1.0000000000000000D+00   2.0000000000000000D+00  -1.0000000000000000D+00  -0.
 -0.                      -0.                      -0.                      -0.
 -0.                      -0.

  1.5258793540564374D+00  -2.5378053350970018D+00   5.8946946649029982D+00  -8.6318331128747795D+00
  8.4962108686067019D+00  -5.6031294091710758D+00   2.3826576278659612D+00  -5.9217041446208113D-01
  6.5495756172839506D-02   0.

ALPHA-BETA COEFFICIENTS ARE

  0.                       0.                      -1.0000000000000000D+00   0.
  0.                       0.                       0.                       0.
  0.                       0.

  9.9999999999999998D-01  -3.7999999998873216D-17   8.3333333333333538D-02   8.3333333333333114D-02
  7.9166666666666770D-02   7.4999999999999994D-02   7.1345899470899458D-02   6.8204365079365082D-02
  6.5495756172839506D-02   0.

ALPHA-BETA-BAR COEFFICIENTS ARE

 -1.0000000000000000D+00   0.                       0.                       0.
  0.                       0.                       0.                       0.

  5.2587935405643746D-01  -1.4860466269841269D+00   2.3967220568783069D+00  -2.3523423721340388D+00
  1.3948040574603175D+00  -4.6117890211640212D-01   6.5495756172839506D-02   0.
```

METHOD: ADAMS

J = 2

```
ORDER   =      9

A-B COEFFICIENTS ARE

  -1.0000000000000000D+00   2.0000000000000000D+00  -1.0000000000000000D+00  -0.
  -0.                      -0.                      -0.                      -0.
  -0.

   6.5495756172839506D-02   9.3641754850088183D-01  -1.7995811287477954D-01   3.9305114638447972D-01
  -3.7936783509700176D-01   2.4374559082892416D-01  -1.0148589065255732D-01   2.4810405643738977D-02
  -2.7086089065255732D-03

ALPHA-BETA COEFFICIENTS ARE

   0.                       0.                      -1.0000000000000000D+00   0.
   0.                       0.                       0.                       0.
   0.

   1.0000000000000000D+00  -1.0000000000000000D+00   8.3333333333333327D-02   2.4199999999984892D-17
  -4.1666666666666890D-03  -4.1666666666665780D-03  -3.6541005291005306D-03  -3.1415343915343914D-03
  -2.7086089065255732D-03

ALPHA-BETA-BAR COEFFICIENTS ARE

  -1.0000000000000000D+00   0.                       0.                       0.
   0.                       0.                       0.

   6.5495756172839510D-02   6.7409060846560846D-02  -1.1063574735449736D-01   1.0437059082892416D-01
  -5.9990906084656086D-02   1.9393187830687831D-02  -2.7086089065255732D-03
```

METHOD W(.7)

J = 2

```
ORDER  =    9

A-B COEFFICIENTS ARE

 -1.0000000000000000D+00   8.7564630456421515D-02   5.0616609860134268D-01   8.5078730487641567D-01
  2.9350329877940556D-01  -6.1031791102949618D-02  -3.3546280938322262D-01  -2.2387773222741319D-01
 -1.1764900000000000D-01

  6.0959920387967380D-02   1.0969512695810990D+00   1.6447314655155853D+00   2.5630893300246886D+00
  1.5368654235069186D+00   1.3545530065583561D+00   3.9165658385364857D-01   1.6477055706762628D-01
  4.7390661505995012D-03

ALPHA-BETA COEFFICIENTS ARE

 -3.0000000000404779D-18   1.5000000000480068D-17  -8.8183166226464894D+00   1.9718838226659372D+01
 -2.1114748425443143D+01   1.3363585024177962D+01  -5.1967789349751149D+00   1.1650697322274132D+00
 -1.1764900000000000D-01

  8.8183166226464893D+00  -2.8537154849305862D+01   4.1568446370656390D+01  -3.6121569968509387D+01
  2.0283183342012313D+01  -7.4300619125339715D+00   1.6777443355438186D+00  -2.0268308627242229D-01
  4.7390661505995012D-03

ALPHA-BETA-BAR COEFFICIENTS ARE

 -1.0000000000000000D+00  -1.9124353695435785D+00  -2.3187046404858143D+00  -1.8741866065516344D+00
 -1.1361652738380490D+00  -4.5917573222741319D-01  -1.1764900000000000D-01

  1.9779798147047340D+01   1.2119392714369917D+01   6.1037187472080791D+00   2.6511341100709301D+00
  7.3541489644069963D-01   1.7424868936882528D-01   4.7390661505995012D-03
```

METHOD: ADAMS

J = 1

```
ORDER  =    10

A-B COEFFICIENTS ARE

 -1.0000000000000000D+00   2.0000000000000000D+00  -1.0000000000000000D+00  -0.
 -0.                      -0.                      -0.                      -0.
 -0.                      -0.                      -0.

  1.5870197861552028D+00  -3.1060692239858907D+00   8.1677502204585538D+00  -1.3935629409171076D+01
  1.6451905313051146D+01  -1.3558823853615520D+01   7.6864539241622575D+00  -2.8652259700176367D+00
  6.3375964506172840D-01  -6.3140432098765432D-02   0.

ALPHA-BETA COEFFICIENTS ARE

  0.                       0.                      -1.0000000000000000D+00   0.
  0.                       0.                       0.                       0.
  0.                       0.                       0.

  9.9999999999999967D-01   6.8800000000358630D-16   8.3333333333333248D-02   8.3333333333332388D-02
  7.9166666666675680D-02   7.4999999999999732D-02   7.1345899470899512D-02   6.8204365079365052D-02
  6.5495756172839512D-02   6.3140432098765432D-02   0.

ALPHA-BETA-BAR COEFFICIENTS ARE

 -1.0000000000000000D+00   0.                       0.                       0.
  0.                       0.                       0.                       0.
  0.

  5.8901978615520244D-01  -1.9280296516754851D+00   3.7226711309523811D+00  -4.5622574955908287D+00
  3.6047191909171076D+00  -1.7871279761904762D+00   5.0747878086419754D-01  -6.3140432098765432D-02
  0.
```

METHOD: ADAMS

J = 2

```
ORDER   =   10

A-B COEFFICIENTS ARE

  -1.0000000000000000D+00   2.0000000000000000D+00  -1.0000000000000000D+00  -0.
  -0.                      -0.                      -0.                      -0.
  -0.                      -0.

   6.3140432098765432D-02   9.5761546516754850D-01  -2.6474977954144621D-01   5.9089836860670194D-01
  -6.7613866843033510D-01   5.4051642416225750D-01  -2.9933311287477954D-01   1.0960207231040564D-01
  -2.3906525573192240D-02   2.3553240740740741D-03

ALPHA-BETA COEFFICIENTS ARE

   0.                       0.                      -1.0000000000000000D+00   0.
   0.                       0.                       0.                       0.
   0.                       0.

   1.0000000000000000D+00  -9.9999999999999999D-01   8.3333333333333298D-02   7.5599999999843693D-17
  -4.1666666666667634D-03  -4.1666666666665966D-03  -3.6541005291005556D-03  -3.1415343915343876D-03
  -2.7086089065255731D-03  -2.3553240740740741D-03

ALPHA-BETA-BAR COEFFICIENTS ARE

  -1.0000000000000000D+00   0.                       0.                       0.
   0.                       0.                       0.                       0.

   6.3140432098765423D-02   8.3896329365079359D-02  -1.6009755291005292D-01   1.8680693342151675D-01
  -1.4242724867724868D-01   6.8854993386243382D-02  -1.9195877425044092D-02   2.3553240740740741D-03
```

METHOD: W(.7)

J = 2

ORDER = 10

A-B COEFFICIENTS ARE

-1.0000000000000000D+00	-6.1243536954357849D-01	5.6746133992083774D-01	1.2051035738973555D+00
8.8905441219289653D-01	1.4442051804263428D-01	-3.7818506315528735D-01	-4.5870169879566902D-01
-2.7436341255918924D-01	-8.2354300000000000D-02		

5.8139643896519189D-02	1.1650057022757099D+00	2.3110674005302198D+00	3.9513045811672464D+00
2.9756731166017285D+00	2.7857136409356712D+00	1.1029404631628498D+00	5.4046011945731516D-01
9.4695967674904173D-02	6.1376227968678421D-03		

ALPHA-BETA COEFFICIENTS ARE

-5.0000000000009884D-17	1.5000000000043355D-16	-1.4991138258499032D+01	3.9694846621173476D+01
-4.9698259081914905D+01	3.7498418438912737D+01	-1.8189033706382269D+01	5.6183637992691829D+00
-1.0155521125591892D+00	8.2354300000000000D-02		

1.4991138258499032D+01	-5.4685984879672508D+01	9.0642367224629967D+01	-9.0504581405925431D+01
5.9766510659377502D+01	-2.6829333590716370D+01	8.0532087091982715D+00	-1.5189822815437909D+00
1.4993457284671475D-01	-6.1376227968678421D-03		

ALPHA-BETA-BAR COEFFICIENTS ARE

-1.0000000000000001D+00	-2.6124353695435785D+00	-3.6574093991663192D+00	-3.4972798548917044D+00
-2.4480958984241931D+00	-1.2544914239140475D+00	-4.3907201255918924D-01	-8.2354300000000000D-02

3.9752986265069995D+01	2.5984993352743192D+01	1.4528067840946609D+01	7.0224469103172722D+00
2.4924990962896640D+00	7.4826492319772703D-01	1.0697121326863986D-01	6.1376227968678421D-03

METHOD: ADAMS

Order = 11

$J = 1$

A-B COEFFICIENTS ARE

-1.0000000000000000D+00	2.0000000000000000D+00	-1.0000000000000000D+00	-0.
-0.	-0.	-0.	-0.
-0.	-0.	-0.	-0.
1.6500924360169152D+00	-3.7167957226030143D+00	1.0916019464235610D+01	-2.1264347392576559D+01
2.9277161784010742D+01	-2.8949131618767035D+01	2.0511710395121853D+01	-1.0193943953423120D+01
3.3820288888387647D+00	-6.7386693071588905D-01	6.1072649861712362D-02	0.

ALPHA-BETA COEFFICIENTS ARE

0.	0.	-1.0000000000000000D+00	0.
0.	0.	0.	0.
0.	0.	0.	0.
9.9999999999999991D-01	5.2999999999300629D-16	8.3333333333332090D-02	8.3333333333334560D-02
7.9166666666665720D-02	7.4999999999998876D-02	7.1345899470900420D-02	6.8204365079364760D-02
6.5495756172839540D-02	6.3140432098765430D-02	6.1072649861712362D-02	0.

ALPHA-BETA-BAR COEFFICIENTS ARE

-1.0000000000000000D+00	0.	0.	0.
0.	0.	0.	0.
0.	0.		
6.5009243601691476D-01	-2.4166108505691843D+00	5.4327053270803267D+00	-7.9823258878467213D+00
7.8798046812369727D+00	-5.2071963684463683D+00	2.2175129769921437D+00	-5.5172163099246433D-01
6.1072649861712362D-02	0.		

Order = 11

METHOD: ADAMS

J = 2

```
A-B COEFFICIENTS ARE

  -1.0000000000000000D+00   2.0000000000000000D+00  -1.0000000000000000D+00  -0.
  -0.                      -0.                      -0.                      -0.
  -0.                      -0.                      -0.

   6.1072649861712362D-02   9.7829328753807920D-01  -3.5779998020883438D-01   8.3903223705307039D-01
  -1.1103729382114799D+00   1.0615975478996312D+00  -7.3356738265592432D-01   3.5773594075677409D-01
  -1.1695672624058041D-01   2.3033146444604778D-02  -2.0677822370530704D-03

ALPHA-BETA COEFFICIENTS ARE

   0.                       0.                      -1.0000000000000000D+00   0.
   0.                       0.                       0.                       0.
   0.                       0.                       0.

   9.9999999999999994D-01  -9.9999999999999974D-01   8.3333333333332840D-02   5.1599999999971729D-16
  -4.1666666666670060D-03  -4.1666666666664972D-03  -3.6541005291006020D-03  -3.1415343915343700D-03
  -2.7086089065255760D-03  -2.3553240740740740D-03  -2.0677822370530704D-03

ALPHA-BETA-BAR COEFFICIENTS ARE

  -1.0000000000000000D+00   0.                       0.                       0.
   0.                       0.                       0.                       0.
   0.

   6.1072649861712160D-02   1.0043858726150378D-01  -2.1799545554753897D-01   3.0260273869648866D-01
  -2.8717200527096361D-01   1.8465079866121532D-01  -7.7093780062530065D-02   1.8897581970498637D-02
  -2.0677822370530704D-03

RUN COMPLETED
```

METHOD W(.7)

$J = 2$

ORDER = 11

A-B COEFFICIENTS ARE

-1.0000000000000000D+00	-1.0164237056358433D+00	1.8791539429726283D-01	1.3812132456494113D+00
1.4716888279333875D+00	6.5024828766999571D-01	-2.4471895668461267D-01	-6.2153414849839357D-01
-5.1115772218685649D-01	-2.3958321254435133D-01	-5.7648010000000000D-02	
5.5882250823470028D-02	1.2108770093303297D+00	2.6897470258232360D+00	5.2993162394060103D+00
4.3946729968204630D+00	4.9672265799196244D+00	2.0811361030535331D+00	1.4877133820205324D+00
3.0439162440683544D-01	9.7874788676236224D-02	1.5648157380348494D-03	

ALPHA-BETA COEFFICIENTS ARE

-2.0000000000026357D-17	9.0000000000608491D-17	-2.2590402816018306D+01	6.8545206393242585D+01
-9.8776156715097821D+01	8.7209898313956176D+01	-5.1138946171130861D+01	2.0253552777589893D+01
-5.2615670850860185D+00	8.1606331254435133D-01	-5.7648010000000000D-02	
2.2590402816018305D+01	-9.1135609209260890D+01	1.6920389667634193D+02	-1.9169815556182421D+02
1.4648606419961178D+02	-7.8468512126645334D+01	2.9568188814379813D+01	-7.6341166581839019D+00
1.2556814307045297D+00	-1.1352294605658472D-01	1.5648157380348494D-03	

ALPHA-BETA-BAR COEFFICIENTS ARE

-1.0000000000000001D+00	-3.0164237056358433D+00	-4.8449320169744238D+00	-5.2922270826635929D+00
-4.2678333204193746D+00	-2.5931912705051605D+00	-1.1632681772755592D+00	-3.5487923254435133D-01
-5.7648010000000000D-02			
6.8601088644066055D+01	4.7277445088201549D+01	2.8643548558160279D+01	1.5308968267525020D+01
6.3690609737102234D+00	2.3963802598150513D+00	5.0483564897341244D-01	1.0100442015230592D-01
1.5648157380348494D-03			

METHOD: ADAMS

J = 1

```
ORDER : =    12

A-B COEFFICIENTS ARE

  -1.0000000000000000D+00   2.0000000000000000D+00  -1.0000000000000000D+00  -0.
  -0.                       -0.                      -0.                      -0.
  -0.                       -0.                      -0.                      -0.
  -0.

   1.7093330001402918D+00  -4.3684419279601571D+00   1.4174250491021324D+01  -3.1039040472933702D+01
   4.8826547944725028D+01  -5.6318272243767035D+01   4.7880851020121853D+01  -2.9743330114137406D+01
   1.3156721969195928D+01  -3.9320979575016033D+00   7.1271885521885522D-01  -5.9240564123376623D-02
   0.

ALPHA-BETA COEFFICIENTS ARE

   0.                       0.                      -1.0000000000000000D+00   0.
   0.                       0.                       0.                       0.
   0.                       0.                       0.                       0.
   0.

   1.0000000000000000D+00  -1.5469999999946114D-15   8.3333333333340835D-02   8.3333333333315595D-02
   7.9166666666692810D-02   7.4999999999751860D-02   7.1345899470914174D-02   6.8204365079359990D-02
   6.5495756172840405D-02   6.3140432098765365D-02   6.1072649861712367D-02   5.9240564123376623D-02
   0.

ALPHA-BETA-BAR COEFFICIENTS ARE

  -1.0000000000000000D+00   0.                       0.                       0.
   0.                       0.                       0.                       0.
   0.                       0.                       0.

   7.0933300014029335D-01  -2.9497759276795719D+00   7.5653656355218868D+00  -1.2958533274210357D+01
   1.5344115760782428D+01  -1.2671507447991822D+01   7.1937203633557806D+00  -2.6843819394340227D+00
   5.9423772697210197D-01  -5.9240564123376623D-02   0.
```

ORDER N = 12

A-B COEFFICIENTS ARE

-1.0000000000000000D+00	2.0000000000000000D+00	-1.0000000000000000D+00	-0.
-0.	-0.	-0.	-0.
-0.	-0.	-0.	-0.

5.9240564123376623D-02	9.9844623065977233D-01	-4.5856469581729998D-01	1.1413263838784672D+00
-1.7149612318622735D+00	1.9080211590107423D+00	-1.5799909937670354D+00	9.6232423440756774D-01
-4.1925087306597723D-01	1.2379786205307039D-01	-2.2220725358746192D-02	1.8320857383357383D-03

ALPHA-BETA COEFFICIENTS ARE

0.	0.	-1.0000000000000000D+00	0.
0.	0.	0.	0.
0.	0.	0.	0.

1.0000000000000000D+00	-1.0000000000000001D+00	8.3333333333337270D-02	-7.5949999999985151D-16
-4.1666666666657410D-03	-4.1666666666674106D-03	-3.6541005291001254D-03	-3.1415343915345390D-03
-2.7086089065255405D-03	-2.3553240740740765D-03	-2.0677822370530707D-03	-1.8320857383357383D-03

ALPHA-BETA-BAR COEFFICIENTS ARE

-1.0000000000000000D+00	0.	0.	0.
0.	0.	0.	0.
0.	0.		

5.9240564123376725D-02	1.1692735890652566D-01	-2.8395054212762539D-01	4.5649794071669077D-01
-5.1801480830126658D-01	4.1549360169151837D-01	-2.3098898208273207D-01	8.4852668550585221D-02
-1.8556553882074715D-02	1.8320857383357383D-03		

METHOD W(.7)

ORDER = 12

J = 2

A-B COEFFICIENTS ARE

-1.0000000000000000D+00	-1.7164237056358433D+00	-5.2358119964782747D-01	1.5127540216574953D+00
2.4385380998879754D+00	1.6804304672233670D+00	2.1045484468438433D-01	-7.9283741817762243D-01
-9.4623162613573198D-01	-5.9739361807515087D-01	-2.2535625878104593D-01	-4.0353607000000000D-02
5.3752967229970020D-02	1.2734167044352588D+00	3.4202503347119664D+00	7.5334709504097768D+00
7.4015307785496677D+00	9.0272266978909519D+00	4.5744656888002667D+00	3.6471722400130080D+00
9.9445919889370690D-01	4.2805952340352144D-01	4.6655048282900124D-02	3.2246546101244020D-03

ALPHA-BETA COEFFICIENTS ARE

4.9999999999902599D-17	-2.5999999999949856D-16	-3.8403684787231119D+01	1.3234013283972521D+02
-2.1590111089093610D+02	2.1740013683429397D+02	-1.4798313731069179D+02	7.0228302041694421D+01
-2.3122150988959157D+01	5.0704045908856102D+00	-6.6924593578104593D-01	4.0353607000000000D-02
3.8403684787231119D+01	-1.7074381762695633D+02	3.5144155079626390D+02	-4.4432959212854051D+02
3.8321501803261697D+02	-2.3593671555502531D+02	1.0519387947309742D+02	-3.3675730488978503D+01
7.4785400929264318D+00	-1.0719660097893648D+00	8.2126248994268546D-02	-3.2246546101244020D-03

ALPHA-BETA-BAR COEFFICIENTS ARE

-9.9999999999999979D-01	-3.7164237056358431D+00	-6.9564286109195140D+00	-8.6836794945456895D+00
-7.9723922782838896D+00	-5.5806745947987227D+00	-2.9785020666291715D+00	-1.1691669566372427D+00
-3.0606347278104593D-01	-4.0353607000000000D-02		
1.3239388580695518D+02	9.5317370691389287D+01	6.1661105910535363D+01	3.5538312080091216D+01
1.6817049028196736D+01	7.1230126741932086D+00	2.0034420089899478D+00	5.3104358379969489D-01
5.3104357503148928D-02	3.2246546101244020D-03		

```
ORDER  =   13

A-B COEFFICIENTS ARE

 -1.0000000000000000D+00   2.0000000000000000D+00  -1.0000000000000000D+00  -0.
 -0.                      -0.                      -0.                      -0.
 -0.                      -0.                      -0.                      -0.
 -0.                      -0.

  1.7669366269298168D+00  -5.0596854495950478D+00   1.7976089860896471D+01  -4.3711838375461681D+01
  7.7340343232037335D+01  -1.0194034471406570D+02   1.0110660224783575D+02  -7.5365402605634004D+01
  4.1670517283005657D+01  -1.6604895877694530D+01   4.5145582321599807D+00  -7.5048408736417169D-01
  5.7603626950115459D-02   0.

ALPHA-BETA COEFFICIENTS ARE

  0.                       0.                      -1.0000000000000000D+00   0.
  0.                       0.                       0.                       0.
  0.                       0.                       0.                       0.
  0.                       0.

  9.9999999999999147D-01   4.6881999999989276D-14   8.3333333333228844D-02   8.3333333334528870D-02
  7.9166666665965050D-02   7.5000000000128520D-02   7.1345899470908336D-02   6.8204365079358172D-02
  6.5495756177841855D-02   6.3140432098764970D-02   6.1072649861712404D-02   5.9240563962786182D-02
  5.7603626950115459D-02   0.

ALPHA-BETA-BAR COEFFICIENTS ARE

 -1.0000000000000000D+00   0.                       0.                       0.
  0.                       0.                       0.                       0.
  0.                       0.                       0.                       0.

  7.6693662692977845D-01  -3.5258121957354440D+00   1.0157528842495805D+01  -1.9870968494734628D+01
  2.7440877400072275D+01  -2.7187621419186523D+01   1.9290482009390430D+01  -9.5968171676666218D+00
  3.1864009382819837D+00  -6.3527683346394077D-01   5.7603626950115459D-02   0.

RUN COMPLETED
```

METHOD: ADAMS

J = 2

ORDER = 13

A-B COEFFICIENTS ARE

-1.0000000000000000D+00	2.0000000000000000D+00	-1.0000000000000000D+00	-0.
-0.	-0.	-0.	-0.
-0.	-0.	-0.	-0.
-0.			

5.7603625860394627D-02	1.0180894898155563D+00	-5.6660262117411175D-01	1.5014528017345064D+00
-2.5252456720383618D+00	3.2044762632924835D+00	-3.0925219487624002D+00	2.2587793386893089D+00
-1.2295353132420655D+00	4.8392427990910961D-01	-1.3025865071555796D-01	2.1475344894119696D-02
-1.6369382629819965D-03			

ALPHA-BETA COEFFICIENTS ARE

0.	0.	-1.0000000000000000D+00	0.
0.	0.	0.	0.
0.	0.	0.	0.
0.			

9.9999999999999983D-01	-9.9999999999999899D-01	8.3333333333330821D-02	3.6500000000010055D-15
-4.1666666666701275D-03	-4.1666666666644640D-03	-3.6541005291014740D-03	-3.1415343915341120D-03
-2.7086089065256375D-03	-2.3553240740740600D-03	-2.0677822370530730D-03	-1.8320857383357380D-03
-1.6369382629819965D-03			

ALPHA-BETA-BAR COEFFICIENTS ARE

-1.0000000000000000D+00	0.	0.	0.
0.	0.	0.	0.
0.	0.	0.	

5.7603625860393788D-02	1.3329674153634489D-01	-3.5761276396181576D-01	6.5293053227452999D-01
-8.6177184352748606D-01	8.2800204396298139D-01	-5.7474601730895136D-01	2.8128526010842479D-01
-9.2218775716264557D-02	1.8201468368155703D-02	-1.6369382629819965D-03	

METHOD W(.7)

J = 2

ORDER = 13

A-B COEFFICIENTS ARE

```
-1.0000000000000000D+00  -1.7196257477382941D+00  -9.6855656785147289D-01   1.0348794299864174D+00
 2.4885564266525799D+00   2.3168794561166209D+00   9.1092128368820963D-01  -4.6805688983022413D-01
-1.0508458630278880D+00  -8.8705933945534025D-01  -4.7091069113873375D-01  -1.5793397250187466D-01
-2.8247524900000000D-02

 5.2621506036380960D-02   1.2866221511014238D+00   3.3801533367459571D+00   8.2976819824392245D+00
 8.1713878096467131D+00   1.2189439038563833D+01   5.8795476201930563D+00   6.6241117655518075D+00
 1.6372424674652942D+00   1.2962117932955695D+00   1.5549002457758930D-01   5.7898226050484721D-02
 2.5948367399967170D-04
```

ALPHA-BETA COEFFICIENTS ARE

```
 4.9999999999979908D-17  -2.8000000000007555D-16  -4.9028667205341333D+01   1.8614886249379838D+02
-3.3896588625507677D+02   3.8667066181275043D+02  -3.0425959906675298D+02   1.7160849359751226D+02
-7.0266991307678288D+01   2.0496990216445784D+01  -4.0725210320593550D+00   4.9690427130187466D-01
-2.8247524900000000D-02

 4.9028667205341334D+01  -2.3517752969913972D+02   5.2920047101598694D+02  -7.4114895327564373D+02
 7.1897313195407088D+02  -5.0771931366829255D+02   2.6661455821630518D+02  -1.0435640467969308D+02
 2.9981851430076755D+01  -6.0926008801280499D+00   8.0949643361689956D-01  -6.1012030138480781D-02
 2.5948367399967170D-04
```

ALPHA-BETA-BAR COEFFICIENTS ARE

```
-9.9999999999999977D-01  -3.7196257477382939D+00  -7.4078080633280610D+00  -1.0061110948931411D+01
-1.0225857407882180D+01  -8.0737244107163292D+00  -5.0106701298622684D+00  -2.4156727388384317D+00
-8.7152121084248307D-01  -2.1442902230187466D-01  -2.8247524900000000D-02

 1.8620148399983477D+02   1.3851206045163124D+02   9.4202790240173666D+01   5.8191202011155319D+01
 3.0351001591783684D+01   1.4700240210975883D+01   4.9290264503611383D+00   1.7819244552982009D+00
 2.7206492770055776D-01   5.8417193398484064D-02   2.5948367399967170D-04
```

5. Cyclic Methods

The investigation of the five and six backpoint cyclic methods derived under contract NAS-51187 was divided into five tasks. Each task includes analysis, computer programming and the running of computer programs. The first two tasks were funded by this contract, the last three by NASA grant NGR 05-071-005 at California State University, Fullerton under Dr. Sam Pierce.

1. Local correction error (algorithm).
2. Local truncation error (order).
3. Eigenvalues (stability).
4. Comparisons (computations).
5. Analysis and conclusions.

1. Local correction error (algorithm). Iterating to convergence on elliptical orbits implied that the predictors used caused instability in some cases. Correcting twice removed this instability. However, correcting more than twice did not significantly improve the computed solution which remains poor.

2. Local truncation error (order). The order equations were satisfied in all cases. The first few non-zero coefficients

in the local truncation error expansion, C_i, are comparable to Cowell's. For example, for the k=5, order 7, cyclic methods [6].

Method #	1	2	3	4	5
C_9	-.0005	.003	-.0003	-.0007	-.0009
C_{10}	-.001	.006	-.0007	-.002	-.002

3. Eigenvalues (stability). The remaining tasks considered the equation y" = y. Since the behavior of the cyclic methods on this equation is similar to that on orbit equations it was felt that improvements on this equation would be reflected on orbit equations. This work should be extended to y" = -y. From equation 2.4(2) in [2]. Let

$$A = -(A_1 - h^2B_1)^{-1}(A_o - h^2B_c).$$

This "stability matrix" describes the error propagation properties of the cyclic method applied to the above equation. The eigenvalues were computed using computer program HESSEN (supplied by Dr. Velez) for $h = 0, 10^{-6}, 10^{-4}$, and 10^{-2}. All methods showed excessive error in computations at $h = 10^{-1}$. The h = 0 eigenvalues were not computed as accurately as the others since the Cowell and cyclic matrices were ill-conditioned, the eigenvectors corresponding to $\lambda = 1$ were equal;

the Jordan canonical form was not diagonal. In the vector (r, i) r is the real part, i the complex.

Cowell K=5 Order 6

h\root #	1	2	3	4	5
0	1	1	-5×10^{-23}	-7×10^{-35}	-3×10^{-35}
10^{-6}	1.000005	0.999995	1×10^{-28}	-8×10^{-28}	-1×10^{-28}
10^{-4}	1.0005	0.9995	1×10^{-18}	$(-2,-4)10^{-18}$	$(-2,+4)10^{-18}$
10^{-2}	1.05	0.95	2×10^{-11}	$(-1,-2)10^{-11}$	$(-1,+2)10^{-11}$

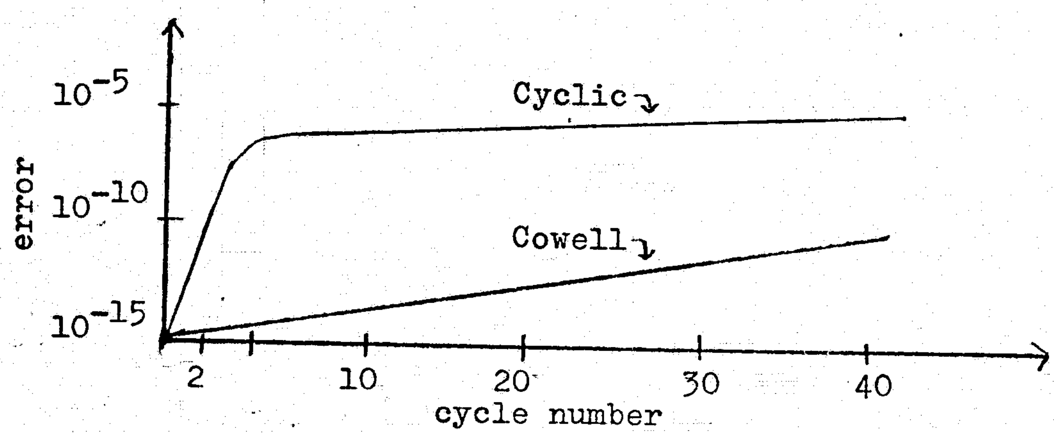
Cyclic K=5 Order 7

h\root #	1	2	3	4	5
0	1	1	-1×10^{-4}	$(.7,1)10^{-4}$	$(.7,-1)10^{-4}$
10^{-6}	1.000005	0.999995	2×10^{-6}	$(-1,2)10^{-6}$	$(-1,-2)10^{-6}$
10^{-4}	1.0005	0.9995	1×10^{-4}	-1×10^{-5}	-9×10^{-5}
10^{-2}	1.05	0.95	2×10^{-2}	-1×10^{-5}	-5×10^{-3}

The principle eigenvalues of the cyclic method behave the same as the Cowell. The Cowell extraneous values remain well within the unit circle when error becomes excessive in computations at $h = 10^{-1}$. One might say Cowell is "over-stable". The cyclic extraneous values exit the unit circle at the same time error growth becomes excessive in computations. Since

the cyclic method is not "over-stable" it may improve if the extraneous root behavior with h is improved.

4. Comparisons (computations). In solving $y''=y$ without using a predictor the Cowell and cyclic method blew up at $h=10^{-1}$. The relative behavior of theCowell and cyclic K=5 methods at $h=10^{-6}$, 10^{-4}, and 10^{-2} was the same and is qualitatively sketched below for $h=10^{-4}$.



The Cowell curve illustrates a stable method. The cyclic curve implies a stable method in the third cycle and beyond, in fact slightly more stable than Cowell since the scope is slightly smaller.

5. Analysis and conclusions. Tasks 1 thru 4 imply there is only one thing wrong with the cyclic methods: the error jump in the first cycles. This is not a stability problem since the latter would be a long range effect. What is the cause of this problem? Since both Cowell and cyclic stability matrices

are degenerate at $h=0$ but not at $h\neq0$ (since eigenvalues are unequal) this cannot be the cause as was originally expected. Since the first cycle error equals A times the initial error, it must be that norm (A) is too large. In fact the elements of the cyclic A are as large as 7000 while Cowell's are less than 6. There are two ways to look at it:

1. The coefficients of the cyclic method are too large.
2. Since $A = PJP^{-1}$, P = the normalized eigenvectors, and J = the eigenvalues then P^{-1} is too large, P is ill-conditioned, and the eigenvectors are nearly dependent.

Methods of correcting this problem must be discovered and analyzed and computer programs written and implemented. Once corrected, the many degrees of freedom available with cyclic methods can be used to improve them above Cowell's. For example, minimizing the first few non-zero C_i, improving the root behavior at $h\neq0$, extending these improvements to other equations including orbit, and including predictors in the analysis.

6. Off-Grid Algorithms with Higher Step Number

Various off-grid difference equations of the form (Class II)

$$\sum_{i=0}^{k} a_i \, y_{n-i} + h^2 \sum_{i=0}^{k} b_i \, \ddot{y}_{n-i} + h^2 \, b_\theta \, \ddot{y}_{n-\theta} = 0 \qquad (6\text{-}1)$$

have been examined for $h = 0$ stability by use of Program A-B on the GSFC 360/95 computer. In all computations θ and b_θ were varied as follows:

$$\theta \; : - .8 \; (.2) \; .8$$

$$b_\theta \; : - .5 \; (\; 1) \; .5 \; .$$

Methods are then specified by giving step-number k and the coefficients a_i, b_i set equal to zero, in the solution of order equations [10] for the remaining coefficients.

Cases studied are listed below. Indices listed in the table below are those of the coefficient c_i arbitrarily set equal to zero. For this purpose, we define

$$c_i = a_i, \qquad i = 1,2,\ldots,k;$$

$$c_{k+i+1} = b_i, \qquad i = 0,1,\ldots,k.$$

The order of the method is $2k + 1$, minus the number of c_i set equal to zero. The indices specifying methods are:

	Order 12	Order 11	Order 10
k = 7	7	6,7	5,6,7
	9	7,9	5,6,9
	10	7,10	5,6,10
	11	7,11	5,6,11
	12	7,12	5,6,12
	13	7,13	5,6,13
	14	7,14	5,6,14
k = 8	6,7,8	5,6,7,8	4,5,6,7,8
	7,8,10	6,7,8,10	5,6,7,8,10
	7,8,11	6,7,8,11	5,6,7,8,11
	7,8,13	6,7,8,13	5,6,7,8,13
	7,8,14	6,7,8,14	5,6,7,8,14
	7,8,15	6,7,8,15	5,6,7,8,15
k = 9	5,6,7,8,9	5,6,7,8,9,14	3,4,5,6,7,8,9
	6,7,8,9,11	6,7,8,9,11,14	4,5,6,7,8,9,11
	6,7,8,9,12	6,7,8,9,12,14	4,5,6,7,8,9,12
	6,7,8,9,13	6,7,8,9,13,14	4,5,6,7,8,9,13
	6,7,8,9,14		4,5,6,7,8,9,14
	6,7,8,9,15	6,7,8,9,15,14	4,5,6,7,8,9,15
	6,7,8,9,16	6,7,8,9,16,14	4,5,6,7,8,9,16

Of the cases examined, stability was encountered only for $k = 9$ and the sets of indices $\{3,4,5,6,7,8,9\}$, $\{4,5,6,7,8,9,12\}$ and $\{4,5,6,7,8,9,14\}$, all of order 10. The first group of methods is, of

course, stable for all b_θ, since the stability polynomial is $x^9 - 2x^8 - x^7$. The remaining groups are stable for some pairs θ, b_θ. Label the groups, Group 1, Group 2 and Group 3, respectively.

<u>Group 1</u>.

The extraneous roots are all zero. The magnitudes of the primary error coefficients C_{12} are in the approximate range

$$.002 \leq |C_{13}| \leq .27 \quad .$$

<u>Group 2</u>.

Stable configurations were found for the pairs, θ, b_θ, as follows:

$$\begin{array}{lll}
\theta = 0 & & b_\theta = -.5 \\
\theta = 0 & & b_\theta = .5 \\
\theta = .2 & & b_\theta = .5 \\
\theta = .6 & & b_\theta = .5 \\
\theta = .8 & & b_\theta = -.5 \\
\theta = .8 & & b_\theta = .5 \; .
\end{array}$$

The magnitudes of the coefficients C_{12} for these methods were in the range

$$.0007 \leq |c_{12}| \leq .03 .$$

The largest extraneous root in these cases satisfied

$$.2 \leq |r_\theta| \leq .98 .$$

<u>Group 3</u>.

Stable configurations were found for the pairs θ, b_θ, as follows:

$$\theta = .8 \qquad b_\theta = .5 .$$

The primary error coefficient is $C_{12} = .0003$ and the extraneous root with largest magnitude is $r_\theta = -.93$.

Program GENCOB was used in a search for stable 7-step methods. As described in [2], coefficients in the off-grid Class II difference equation (6-1) are determined by the second-derivative of quasi-Hermite interpolating polynomial P, evaluated at $t = k - \theta$. This polynomial is explicitly computed by requiring certain conditions, $C(i)$, on P and $\ddot{P}$, at points of our interpolating set. A definition of the conditions, with their indexing, will describe what methods have been examined for stability.

Interpolation is on a set, $\{t_n, t_{n-1}, \ldots, t_{n-k}\}$, where k is

the step-number of the method. A sufficiently differentiable function, $y(t)$, is here being interpolated on the set. The conditions $C(i)$ are defined as:

$$C(i): \quad P(t_{n-i}) = y(t_{n-i}) \quad , \quad i = 1,2,\ldots,k \quad ;$$

$$C(i): \quad \ddot{P}(t_{n+k-i+1}) = \ddot{y}(t_{n+k-i+1}), \quad i = k+1,\ldots,2k+1 \; .$$

Methods are then determined by specifying k and the conditions omitted from the $2k+1$ possible ones. The polynomial degree of the method is $2k+1$, diminished by the number of conditions omitted. Indexing of conditions corresponds to the indexing of coefficients explained above.

Program GENCOB is capable of deriving methods in which b_θ is essentially determined by order equations. Such methods cannot be obtained by Program A-b. For this reason, the following methods were derived by GENCOB. Stability was examined for values of θ: .2 (.05) .5

$k = 7$

Conditions Eliminated	Order
None	15
C(8)	14
C(9)	13
C(10)	13
C(11)	13
C(12)	13
C(13)	13

Conditions Eliminated	Order
c(8), c(9)	12
c(8), c(10)	12
c(8), c(11)	12
c(8), c(12)	12
c(8), c(13)	12
c(9), c(10)	12
c(9), c(11)	12
c(9), c(12)	12
c(9), c(13)	12
c(10), c(11)	12
c(10), c(12)	12
c(10), c(13)	12
c(11), c(12)	12
c(11), c(13)	12
c(12), c(13)	12
c(8), c(9), c(10)	11
c(8), c(9), c(11)	11
c(8), c(9), c(12)	11
c(8), c(9), c(13)	11
c(7), c(9), c(14)	11
c(8), c(10), c(11)	11
c(8), c(10), c(12)	11
c(8), c(10), c(13)	11
c(8), c(11), c(12)	11
c(8), c(11), c(13)	11
c(7), c(8), c(9)	11
c(7), c(8), c(10)	11
c(7), c(8), c(11)	11
c(7), c(9), c(10)	11
c(7), c(9), c(11)	11
c(7), c(10), c(11)	11

Of these methods, stability at $h = 0$ were found in the following instances. The approximate range of θ is also given.

Method	Approximate Stable Range of θ	Magnitude of Primary Error Coefficient $\theta = .3$
C(9), C(10)	$.25 < \theta < .35$	1.7×10^{-3}
C(10), C(11)	$.20 < \theta < .35$	7.5×10^{-4}

The following methods may be found to be stable if examined with a finer θ-grid.

C(9), C(11)

C(9), C(12)

C(9), C(13) .

Plots of the magnitude of the largest extraneous root in the difference equation (6-1), as a function of θ, were made at GSFC for various configurations of conditions imposed on the corresponding interpolating polynomial. These plots display many stable Class I and Class II methods. Programs GENCOA and GENCOB were used for this purpose. These results have been included in the first interim deliverable.

7. Methods Using Third Derivative Values and Repeated Off-Grid Accelerations

7.1 Motivation

The usefulness of methods derived by previous coefficient generators GENCOA, GENCOB and Program A-B, have motivated the extension of the latter program to include the capability of generating methods which use more than one off-grid derivative value $y_{n-\theta-i}$ and also, methods which use third derivative values at grid points.

The original off-grid methods use a single off-grid derivative value $\ddot{y}_{n-\theta} = f(y^*_{n-0}, \dot{y}^*_{n-\theta}, t)$, where the starred quantities are predicted values. Since this derivative value is never reused, it need not be corrected. Presently investigated methods use the derivative value in subsequent computations (as n increases) and the off-grid point is corrected once. The new coefficient generator is designed to produce off-grid correctors and all necessary predictors. Use of more off-grid points permits methods of much higher order for the same step-number in comparison with the original off-grind method.

Methods using third derivative values are suggested by the availability of this derivative in some modes of operation of the system, e.g., if the equations of motion are $\ddot{y} = F(y, \dot{y}, t)$, the partial derivatives $\partial F/\partial y$ and $\partial f/\partial \dot{y}$ are available when variational equations are being solved, permitting immediate computation of the third derivative. Possible advantages in accuracy of these methods are

examined. Use of the third derivative extends the scope of this task.

The coefficient generator ABX

The coefficient generator ABX is designed to compute the coefficients in a method of the general form

$$\sum_{i=0}^{k} (a_i\, y_{n-i} + h^2\, b_i\, \ddot{y}_{n-i} + h^3\, c_i\, \dddot{y}_{n-i} + h^2\, d_i\, \ddot{y}_{n-\theta-i} + e_0\, y_{n-\theta}) = 0 \;.$$

A grid for θ can be specified, if desired, or a grid for the pair, θ, d_0, can be specified. This capability was found useful in Program A-B in searching for stable methods. The grids can consist of a single point, or a single pair.

The order equations are produced and solved in a manner similar to that of the parent program, Program A-b. Stability at $h = 0$ is examined and the first three error coefficients are computed. The magnitude of the largest non-principle root is printed.

In addition to the above capabilities, stable methods found are automatically evaluated by integration of a circular trajectory for a specified interval and the error printed. In connection with the integration, it is necessary to use appropriate auxiliary difference equations. For methods using more than one off-grid point, we have

used an on-grid predictor, an off-grid predictor and an off-grid corrector. For third derivative methods, the off-grid corrector is not used. As a standard of comparison, the old off-grid method, $\theta = .21$, is processed along with the new methods. Because the integration procedure is different for third derivative methods, there exist separate programs: Program ABXMO generates and tests methods which need no third derivative information and use more than one off-grid point, while Program ABXJRK handles methods using only a single off-grid and may or may not require third derivatives. A few mixed cases have been generated which involve more than one off-grid acceleration as well as third derivative values. These have been run in Porgram ABXJRK. Methods tested by Program ABXMO should be compared with each other. The same is true of methods tested with ABXJRK. Methods from one group should not be compared to those of the other. Methods from ABXMO are integrated with a correction of the off-grid acceleration. ABXJRK does no correction of the off-grid acceleration. A description of computer programs is found in 7.4.

Methods Generated and Tested

Methods processed are given below. In case the method is stable an integration is performed as described above. The additional precision of the CDC 6400 was needed in the computation of difference equation coefficients. In view of this fact, all computations, including the numerical integration, were done on that machine.

Generated methods are defined below. In all cases, $e_0 = 0$, $a_0 = -1$. In case the method contains more than one off-grid point, an off-grid corrector is generated at the time of integration. This auxiliary corrector has the same specifications, except that $e_0 = -1$, $a_0 = 0$. These methods are produced by program ABXMO. In off-grid cases, θ was varied on the grid, i.e., θ: $-.6\ (.4)\ .6$. For the meaning of coefficients a_i, b_i, c_i and d_i refer to the general difference equation above. Methods are defined by specifying coefficients constrained to be zero. For each group of methods, the right-hand column indicates which deck was used to generate and test the methods.

	Order	Coefficients Constrained to Equal Zero		Program
$k = 3$	$e_0 = 0$	$c_j = 0,$	$j = 0,1,2,3$	ABXMO
1.	9		--	
2.	8		d_1	
3.	8		d_3	
4.	8		a_1	
5.	8		a_3	
$k = 3$	$e_0 = 0$	$d_j = 0,$	$j = 1,2,3$	ABXJRK
1.	10		--	

	Order	Coefficients constrained to Equal Zero	Program
$k = 4$	$e_0 = 0$	$c_j = 0, \quad j = 0,1,2,3,4$	ABXMO
1.	12	--	
2.	11	d_1	
3.	11	d_2	
4.	11	d_4	
5.	10	d_1, d_2	
6.	10	d_2, d_3	
7.	9	d_2, d_3, d_4	
8.	11	a_0	
9.	11	a_2	
10.	11	a_4	
11.	10	a_3, a_4	
$k = 6$	$e_0 = 0$	$c_j = 0, \quad j = 0,1,\ldots,6$	ABXMO
1.	14	d_1, d_2, d_3, d_4	
2.	13	d_2, d_3, d_4, d_5, d_6	
3.	11	$a_3, a_4, a_5, a_6 \qquad d_4, d_5, d_6$	
$k = 5$	$e_0 = 0$	$c_j = 0, \quad j = 0,1,2,3,4,5$	ABXMO
1.	13	d_1, d_2	
2.	12	d_1, d_2, d_3	
3.	11	d_1, d_2, d_3, d_4	
4.	11	d_2, d_3, d_4, d_5	

k = 4 (cont'd)

5.	11	a_3,a_4,a_5,d_2
6.	11	a_3,a_4,a_5,d_1
7.	11	a_4,a_5,d_4,d_5
8.	11	a_3,a_4,a_5,d_5

k = 4	$e_0 = 0$	$d_j = 0, \quad j = 1,2,3,4$	ABXJRK
1.	13	--	
2.	11	c_0,c_1	
3.	11	b_0,b_1	
4.	11	c_1,c_2	
5.	11	b_1,b_2	
6.	11	a_3,a_4	

k = 5	$e_0 = 0$	$d_j = 0, \quad j = 1,2,3,4,5$	ABXJRK
1.	11	a_3,a_4,a_5,c_0,c_1	
2.	11	a_3,a_4,a_5,b_0,b_1	
3.	11	a_3,a_4,a_5,c_1,c_2	
4.	11	a_3,a_4,a_5,b_1,b_2	
5.	11	a_3,a_4,a_5,c_2,c_3	
6.	11	a_3,a_4,a_5,b_2,b_3	
7.	11	a_2,a_3,a_4,b_2,c_2	
8.	11	a_3,a_4,a_5,b_5,c_5	

$k = 6$	$e_0 = 0$	$d_j = 0, \quad j = 1,2,\ldots,6$	ABXJRK
1.	11	a_3,a_4,a_5,a_6 c_3,c_4,c_5,c_6	
$k = 4$	(no off-grid point) $e_0 = 0$	$d_j = 0, \quad j = 0,1,\ldots,4$	ABXJRK
1.	12	--	
2.	11	c_0	
3.	11	c_1	
4.	11	c_2	
5.	10	c_1,c_2	
6.	10	c_3,c_4	
7.	10	a_3,a_4	
$k = 3$	(mixed methods) $e_0 = 0$		ABXJRK
1.	11	d_2,d_3	
2.	11	d_1,d_3	

Methods are easily identified and evaluated. The off-grid second derivatives are positioned by θ, e.g., $\theta = .2$ means terms such as $h^2 d_i \ddot{y}_{n-i-.2}$ which appear in the difference equation for non-zero d_i. Printed coefficients are indexed from current time, for example, in the order $a_0 = -1$, $a_1, a_2, \ldots$, etc. Coefficients either appearing as zeros (or missing) have been arbitrarily set to zero. Indexed error coefficients are given. Stability (at $h = 0$) is noted, as well as the magnitude of the largest extraneous root. Finally, the result of an integration of a circular trajectory is given. These comprise the final two lines of the output. The first of these lines shows, respectively, time, true x_1 component of position, true x_2 component of position, computed x_1 component of position, computed x_2 component of position. The last line indicates relative error in position, error in the first component, error in the second component, $\ddot{x}_1$, $\ddot{x}_2$, respectively. Basic units are, of course, meters and seconds. In every case, the step-size for the integration is indicated on the output.

It is important to note that in the integration, an evaluation is doen after every prediction or correction . Also, in methods using a third derivative, that derivative is computed assuming that $\dot{r} = 0$. Furthermore, the analytical formula for $\dot{y}$ is used.

Although methods can be identified as described above, input parameters are included in the output. These are described in the program descriptions. See Section 7.2 for a complete description of output.

The methods defined above were initially tested by integration of one revolution of a two-hour circular trajectory. The results of this initial evaluation comprised the first interim deliverable. Although the computer programs, with slight modifications, are capable of producing any type of predicting method, exact (analytical) predictors were used in the preliminary evaluation.

On the basis of the preliminary screening, many methods were reevaluated by an 84 revolution computation of the same trajectory, using a fixed 11^{th} order quasi-Hermite interpolating polynomial for prediction. This polynomial is the basis of the predictors in the original, $\theta = .21$, off-grid method.

The computer output is given below for methods found to be stable at step-sizes greater than 200 secs.

ABXJRK H=200 SECS.

```
K =  4
A-ZERO IS -1.  (IAZEZ= 1)
INDICES FOR DROPPED D COEFS..   1  2  3  4
 THETA START =    5.0000000000000000D-01,   THETA STEP =    1.0000000000000000D-01,   THETA STOP =    6.0000000000000000D-01
```

H=200 SECS.

```
K =   4
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


THETA =      5.0000000000000000D-01
TRUNC COEFS =     1.0592522262798390D-11    9.9096565328712270D-12   -2.2914054186609777D-12,  DET =     6.5549672794946119D+32
TRUNC INDICES =               15                          16                          17
A COEFS ARE..
 -1.0000000000000000D+00    2.4327631238352039D+00   -1.8348841340449850D+00    4.7147889650338535D-01
 -1.3357385333409374D-02
B COEFS ARE..
  2.4032723526987759D-02    4.9193526908437480D-01   -1.5506434706405087D-01   -3.3434663292565107D-02
  1.0805056455590573D-03
C COEFS ARE..
 -1.7157800092497460D-03   -1.8754867530427520D-02    5.3799115958060300D-02   -1.1209197003808596D-02
  1.1056226366566836D-04
D COEFS ARE..
  2.5830652745906531230D-01  0.                        0.                        0.
  0.
STABLE                                                                                                   STABLE
STABILITY =      3.8210148531092994D-01

          T-E                    X1-E1                    X2-E2                    U1-UDD1                   U2-UDD2
  5.9820000000000000D+05    7.9969654990128668D+06    2.2032432366302644D+05    7.9969654990218369D+06    2.2032432338888388D+05
  3.4280378103155295D-11   -3.9500985274657216D-06    2.7414256408824154D-04   -6.2258125746055304D+00   -1.7152730535273039D-01

ORBIT COMPLETED.
```

H=200 SECS.

```
K =  4
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


THETA =     6.0000000000000000D-01
TRUNC COEFS =   -1.9160712045118765D-10  -4.1195781841020363D-10  -4.5538177440451958D-10,  DET =     4.0909181440488437D+32
TRUNC INDICES =              15                       16                       17
A COEFS ARE..
 -1.0000000000000000D+00   1.9532784212361664D+00  -8.8261029510695930D-01  -9.4614673494580749D-02
  2.3946547365373598D-02
B COEFS ARE..
  3.2778707798045908D-02   4.2967087474486489D-01   1.4902858671110030D-01  -8.7207146791259640D-04
 -1.5346788782283911D-03
C COEFS ARE..
 -2.7283334244874962D-03  -9.6765313287554272D-02   3.6269157670401619D-02   7.0714780612130896D-03
 -1.7521570635181554D-04
D COEFS ARE..
 [illegible].4350361249053885D-01   0.                       0.                       0.
  0.
STABLE                                                                                                   STABLE
STABILITY =     1.7906086556694492D-01

           T-E                      X1-E1                      X2-E2                      U1-UDD1                    U2-UDD2
  5.9820000000000000D+05   7.9969654990128868D+06   2.2032432366302644D+05   7.9969654990210748D+06   2.2032432346390400D+05
  2.4911339602991733D-11  -8.1679909435138432D-06   1.9912244126794532D-04  -6.2258125746018920D+00  -1.7152730541105139D-01

ORBIT COMPLETED.
```

ABXJRK   H=200 SECS.

```
K =  4
A-ZERO IS -1.   (IAZEZ= 1)
INDICES FOR DROPPED D COEFS..   1  2  3  4
 THETA =     5.0000000000000000D-01
```

H=200 SECS.

```
K =  4
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


THETA =     5.00000000000000000D-01
TRUNC COEFS =     1.05925272627983900-11   9.9096565328712270D-12  -2.2914054186609777D-12,  DET =     6.5549672794946119D+32
TRUNC INDICES =              15                         16                         17
A COEFS ARE..
 -1.00000000000000000D+00   2.4327631238352039D+00  -1.8848841340044965D+00   4.7147889650338535D-01
 -1.3357686334090740D-02
B COEFS ARE..
  2.4032723526937759D-02   4.9193526908437430D-01  -1.5506434706405087D-01  -3.3434663292565107D-02
  1.0805056455590573D-03
C COEFS ARE..
 -1.7157500097427460D-03  -1.3754667580427520D-02   5.3799115958060300D-02  -1.1209197003808596D-02
  1.1956226366585836D-04
D COEFS ARE..
  2.5806527459858123D-01   0.                        0.                        0.
  0.
STABLE                                                                                                  STABLE
STABILITY =     3.8210148331092994D-01

           T-E                     X1-E1                    X2-E2                    U1-UDD1                  U2-UDD2
  1.5000000000000000D+03   1.9582031093349730D+06   7.7566338484716371 5D+06  1.9582031093350252D+06   7.7566334847168697D+06
  6.9126039503212750D-15  -5.2193506646432000D-08  -1.8276203957897139D-08  -1.5245039563103164D+00  -6.0387127368186828D+00

           T-E                     X1-E1                    X2-E2                    U1-UDD1                  U2-UDD2
  1.6000000000000000D+03   1.2670748590343032D+06   7.8990202747937798D+06   1.2670748590347742D+06   7.8990202747934562D+06
  7.1428956921533018D-14  -4.7096787821721979D-07   3.2360994132662043D-07  -9.8644549502120368D-01  -6.1495600749957616D+00
UDDD=    5.4259958645730011D-03  -8.7037919975261232D-04

           T-E                     X1-E1                    X2-E2                    U1-UDD1                  U2-UDD2
  1.5000000000000000D+03   1.9582031093349730D+06   7.7566338484716871 5D+06  1.9582031093350252D+06   7.7566334847168897D+06
  6.9126039503212750D-15  -5.2193506646432000D-08  -1.8276203957897139D-08  -1.5245039563103164D+00  -6.0387127368186828D+00

           T-E                     X1-E1                    X2-E2                    U1-UDD1                  U2-UDD2
  1.6000000000000000D+03   1.2670748590343032D+06   7.8990202747937798D+06   1.2670748590343025D+06   7.8990202747937801D+06
  9.1971391413234901D-17   6.3569304412614760D-10  -2.6028910920745497D-10  -9.8644549502074583D-01  -6.1495500749954665D+00
UDDD=    5.4259958645725024D-03  -8.7037919975253233D-04

           T-E                     X1-E1                    X2-E2                    U1-UDD1                  U2-UDD2
  5.9820000000000000D+05   7.9969654990126868D+06   2.2032432366302644D+05   7.9969654990213369D+06   2.2032432336888368D+05
  3.4286072103156293D-11  -8.9500985274657216D-05   2.7414256406824154D-04  -6.2258125746055304D+00  -1.7152730535273039D-01

ORBIT COMPLETED.
```

ABXJRK H=200 SECS.

```
K =  5
A-ZERO IS -1.   (IAZEZ= 1)
INDICES FOR DROPPED A COEFS..   3  4  5
INDICES FOR DROPPED C COEFS..   1  2
INDICES FOR DROPPED D COEFS..   1  2  3  4  5
 THETA START =   -6.0000000000000000D-01,   THETA STEP =    4.0000000000000000D-01,   THETA STOP =    6.0000000000000000D-01
```

H=200 SECS.

```
K =  5
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

THETA =     2.0000000000000000D-01
TRUNC COEFS =   -2.5365205514916763D-07   -6.1708877970451928D-07   -7.7183362378317562D-07,   DET =    4.9407269808840583D+31
TRUNC INDICES =          13                         14                        15
A COEFS ARE..
 -1.0000000000000000D+00   2.0000000000000000D+00   -1.0000000000000000D+00    0.
  0.                       0.
B COEFS ARE..
 -1.4699281305114638D-01   6.8536277331850249D-01    2.1176296697130030D-01   -2.4024787885332103D-02
 -3.7585835290867509D-02  -3.3665925329685746D-03
C COEFS ARE..
  1.3274080085530087D-02   0.                        0.                       -6.2421622689479832D-02
 -1.7886058264347738D-02  -7.3998166185666186D-04
D COEFS ARE..
  3.1494435847051188D-01   0.                        0.                        0.
  0.                       0.
STABLE
STABILITY =     0.                                                                         STABLE

          T-E                      X1-E1                     X2-E2                    U1-UDD1                   U2-UDD2
  5.9820000000000000D+05   7.9969654990128868D+06   2.2032432366302644D+05   7.9969654990143863D+06   2.2032432361919846D+05
  5.4817834246083205D-12  -1.4994982333899470D-06   4.3827983536905575D-05  -6.2258125746023092D+00  -1.7152730553210639D-01

ORBIT COMPLETED.
```

ABXJRK H=200 SECS.

```
K =  5
A-ZERO IS -1.  (IAZEZ= 1)
INDICES FOR DROPPED A COEFS..   3  4  5
INDICES FOR DROPPED C COEFS..   3  4
INDICES FOR DROPPED D COEFS..   1  2  3  4  5
 THETA START =   -6.0000000000000000D-01,   THETA STEP =    4.0000000000000000D-01,   THETA STOP =    6.0000000000000000D-01
```

H=200 SECS.

```
K =  5
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


THETA =     6.0000000000000000D-01
TRUNC COEFS =     1.9255419237381652D-08    5.4402944727319395D-08    7.8722479854334053D-08,  DET =     5.8651173820859349D+30
TRUNC INDICES =               13                        14                        15
A COEFS ARE..
 -1.0000000000000000D+00    2.0000000000000000D+00   -1.0000000000000000D+00    0.
  0.                        0.
B COEFS ARE..
  3.0454636644219978D-02    4.1773094286115119D-01    1.1724624975050145D-01    2.0188241542408209D-03
 -1.9937589707683747D-04    3.0282835405080584D-05
C COEFS ARE..
 -2.3394059644059644D-03   -1.0654949795574796D-01    2.6148818113103827D-02    0.
  0.                        8.3751475796930342D-06
D COEFS ARE..
  4.3271743961155831D-01    0.                        0.                        0.
  0.                        0.
STABLE
STABILITY =     0.                                                                                      STABLE

          T-E                      X1-E1                     X2-E2                    U1-UDD1                   U2-UDD2
  5.9820000000000000D+05    7.9969654990128868D+05    2.2032432366302644D+05    8.0118066052529069D+06   -1.2832343548951925D+05
  4.3620436436899740D-02   -1.4841106240020072D+04    3.4864775915254569D+05   -6.2074433979791731D+00    9.9423326308595756D-02

ORBIT COMPLETED.
```

ABXJRK H=200 SECS.

```
K =  5
A-ZERO IS -1.  (IAZEZ= 1)
INDICES FOR DROPPED A COEFS..   3  4  5
INDICES FOR DROPPED B COEFS..   5
INDICES FOR DROPPED C COEFS..   5
INDICES FOR DROPPED D COEFS..   1  2  3  4  5
 THETA START =    -6.0000000000000000D-01,   THETA STEP =    4.0000000000000000D-01,   THETA STOP =    6.0000000000000000D-01
```

H=200 SECS.

```
K =  5
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


THETA =     6.0000000000000000D-01
TRUNC COEFS =     3.5933327179119306D-09    1.0612787125433468D-08    1.5756726890885920D-08,  DET =    -9.8882970739300059D+26
TRUNC INDICES =              13                        14                        15
A COEFS ARE..
 -1.0000000000000000D+00    2.0000000000000000D+00  -1.0000000000000000D+00    0.
  0.                        0.
B COEFS ARE..
  3.2254624340851193D-02    4.2329461947517503D-01   1.2066501384103425D-01    6.0838349322145618D-03
  2.3963519148664285D-04    0.
C COEFS ARE..
 -2.6397824140879696D-03   -9.7998375821709160D-02   3.2819908266336838D-02    2.0824932630538186D-03
  4.8269518245008441D-05    0.
D COEFS ARE..
  4.1746227221923832D-01    0.                       0.                        0.
  0.                        0.
STABLE
STABILITY =     0.                                                                                  STABLE

          T-E                      X1-E1                      X2-E2                     U1-UD01                    U2-UD02
  5.9820000000000000D+05    7.9969654990128868D+06   2.2032432366302644D+05   7.9975485299146939D+06   2.0948038447797735D+05
  1.3574501722065217D-03   -5.8303090180705391D+02   1.0843939185049086D+04  -6.2255658543095686D+00  -1.6306723410096471D-01

ORBIT COMPLETED.
```

ABXJRK H=200 SECS.

```
K =  6
A-ZERO IS -1.   (IAZEZ= 1)
INDICES FOR DROPPED B COEFS..    3
INDICES FOR DROPPED C COEFS..    0  1  2  3  4  5  6
INDICES FOR DROPPED D COEFS..    1  2  3  4  5  6
 THETA =     2.1000000000000000D-01
```

H=200 SECS.

```
K =  6
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

THETA =     2.1000000000000000D-01
TRUNC COEFS =    -2.9134206529425366D-06   -9.3105448146849610D-06   -1.5371615571104099D-05,   DET =    -1.6970863288511077D+30
TRUNC INDICES =               13                            14                            15
A COEFS ARE..
  -1.0000000000000000D+00    1.7567536357760750D+00    -4.5169952884974443D-01    -8.4092156533096416D-01
   9.5197058005475033D-01   -3.5734652759500570D-01    -5.8556594052111089D-02
B COEFS ARE..
  -3.2061504003751415D-02    7.5331041763961414D-01     3.3115350339711752D-01     0.
   3.3456303199945429D-01    9.7061071561566178D-02     2.6019837787390095D-03
D COEFS ARE..
   1.7802152694303854D-01    0.                         0.                         0.
   0.                        0.                         0.
STABLE
STABILITY =      6.6192411214675957D-01                                                                      STABLE

          T-E                         X1-E1                        X2-E2                        U1-UDD1                        U2-UDD2
   1.5580000000000000D+03     1.5588616470340257D+06     7.8466521756355562D+06     1.5588616470341989D+06     7.8466521756355017D+06
   2.3100737771737835D-14    -1.7319517325628295D-07     6.4472113654170040D-08    -1.2136078923146662D+00    -6.1087903642486477D+00

          T-E                         X1-E1                        X2-E2                        U1-UDD1                        U2-UDD2
   1.5000000000000000D+03     1.2670743590343032D+06     7.8990202747937798D+06     1.2670748590347742D+06     7.8990202747934562D+06
   7.1428556921530016D-14    -4.7096787821721979D-07     3.2360994182662043D-07    -9.8644549502120366D-01    -6.1495600749957816D+00
UDDD=    5.4259958648730011D-03    -8.7037919975261232D-04

          T-E                         X1-E1                        X2-E2                        U1-UDD1                        U2-UDD2
   1.5580000000000000D+03     1.5588616470340257D+06     7.8466521756355562D+06     1.5588616470341989D+06     7.8466521756355017D+06
   2.3100737771737835D-14    -1.7319517325628295D-07     6.4472113654170040D-08    -1.2136078923146662D+00    -6.1087903642486477D+00

          T-E                         X1-E1                        X2-E2                        U1-UDD1                        U2-UDD2
   1.5000000000000000D+03     1.2670748590343032D+06     7.8990202747937798D+06     1.2670748590343057D+06     7.8990202747937778D+06
   3.9382410555487962D-16    -2.4925710636344187D-09     1.9917259116981860D-09    -9.8644549502074894D-01    -6.1495600749954708D+00
UDDD=    5.4259958648725059D-03    -8.7037919975253289D-04

          T-E                         X1-E1                        X2-E2                        U1-UDD1                        U2-UDD2
   5.3620000000000000D+05     7.9969654990128665D+06     2.2032432366302644D+05     7.9969654981784304D+06     2.2032435139707749D+05
   3.4693252111530481D-09     5.3445638526017221D-04    -2.7734051053561721D-02    -6.2258125741163766D+00    -1.7152732716233540D-01

ORBIT COMPLETED.
```

ABXJRK H=300 SECS.

```
K =  5
A-ZERO IS -1.  (IAZEZ= 1)
INDICES FOR DROPPED A COEFS..   3  4  5
INDICES FOR DROPPED B COEFS..   2  3
INDICES FOR DROPPED D COEFS..   1  2  3  4  5
 THETA START =   -6.0000000000000000D-01,   THETA STEP =    4.0000000000000000D-01,   THETA STOP =    6.0000000000000000D-01
```

H=300 SECS.

```
K =  5
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

THETA =     6.0000000000000000D-01
TRUNC COEFS =     6.0848155071639798D-07    1.6013329509150756D-06    2.1816105264239839D-06,  DET =    6.1949471124163911D+31
TRUNC INDICES =          13                          14                          15
A COEFS ARE..
 -1.0000000000000000D+00    2.0000000000000000D+00   -1.0000000000000000D+00    0.
  0.                        0.
B COEFS ARE..
 -2.5301188973063973D-03    2.7807041771685522D-01    0.                        0.
  4.3816337484563047D-02    4.8121615170298022D-03
C COEFS ARE..
  3.3889855921115921D-03   -2.1928079154641655D-01    1.3979828042328042D-02    6.0957040644540645D-02
  2.2766939793594205D-02    1.0832930222134768D-03
D COEFS ARE..
  6.7583120217685833D-01    0.                        0.                        0.
  0.                        0.
STABLE
STABILITY =     0.                                                                               STABLE

         T-E                       X1-E1                     X2-E2                     U1-UDD1                   U2-UDD2
  5.9820000000000000D+05    7.9969654990128863D+06    2.2032432366302644D+05    7.9969654953445289D+06    2.2032420844046720D+05
  1.4410114936389315D-08    3.6683579014363871D-03    1.1522253924047932D-01   -6.2258125877157346D+00   -1.7152721630296274D-01

ORBIT COMPLETED.
```

ABXJRK H=300 SECS.

```
K =  6
A-ZERO IS -1.   (IAZE?= 1)
INDICES FOR DROPPED B COEFS..   3
INDICES FOR DROPPED C COEFS..   0  1  2  3  4  5  6
INDICES FOR DROPPED D COEFS..   1  2  3  4  5  6
 THETA =    2.1000000000000000-01
```

3.

H=300 SECS.

```
K =  6
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

THETA =      2.1000000000000000D-01
TRUNC COEFS =    -2.9134206529425366D-06   -9.3105448146849610D-06   -1.5371615571104099D-05,   DET =    -1.6970863288511077D+30
TRUNC INDICES =            13                           14                          15
A COEFS ARE..
 -1.0000000000000000D+00   1.7567536357760750D+00   -4.5189952884974443D-01   -8.4092156533096416D-01
  9.5197056005475033D-01  -3.5734652759800570D-01   -5.8556594052111099D-02
B COEFS ARE..
 -3.2081604003751415D-02   7.5331041763961414D-01    3.3115350339711752D-01    0.
  3.8456803195946429D-01   9.7081071561566178D-02    2.6019837787390995D-03
D COEFS ARE..
  1.7802152594306654D-01   0.
  0.                       0.                        0.                        0.
STABLE                                               0.
STABILITY =     8.6192411214675957D-01
                                                                                                 STABLE
          T-E                    X1-E1                      X2-E2                    U1-UDD1                    U2-UDD2
  2.3370000000000000D+03  -3.7736836154520106D+06   7.0540280670315625D+06  -3.7736836154244015D+06   7.0540280670202984D+06
  3.7273099726068383D-12  -2.7609096751775761D-05   1.1264380654555259D-05   2.9378952439622744D+00  -5.4917151570865431D+00

          T-E                    X1-E1                      X2-E2                    U1-UDD1                    U2-UDD2
  2.4000000000000000D+03  -4.1597676756583158D+06   6.8334715104804683D+06  -4.1597676755611842D+06   6.8334715104406338D+06
  1.3122610198297451D-11  -9.7131528742409281D-05   3.9834502183371915D-05   3.2384701304449422D+00  -5.3200070532359650D+00
UDDD=    4.6940489921371222D-03   2.8574280635400780D-03

          T-E                    X1-E1                      X2-E2                    U1-UDD1                    U2-UDD2
  2.3370000000000000D+03  -3.7736836154520106D+06   7.0540280670315625D+06  -3.7736836154244015D+06   7.0540280670202984D+06
  3.7273099726068383D-12  -2.7609096751775761D-05   1.1264380654555259D-05   2.9378952439622744D+00  -5.4917151570865431D+00
          T-E                    X1-E1                      X2-E2                    U1-UDD1                    U2-UDD2
  2.4000000000000000D+03  -4.1597676756563158D+06   6.8334715104804683D+06  -4.1597676756576254D+06   6.8334715104801087D+06
  9.7306505652352661D-14  -6.9039667050235097D-07   3.5963874105084119D-07   3.2384701304181754D+00  -5.3200070530993854D+00
UDDD=    4.6940489919894967D-03   2.8574280634502133D-03

          T-E                    X1-E1                      X2-E2                    U1-UDD1                    U2-UDD2
  5.9820000000000000D+05   7.9969654990128869D+06   2.2032432366302644D+05   7.9969654149528831D+06   2.2032711715601738D+05
  3.4934468058563175D-07   8.4956003708295826D-02  -2.7934929909343059D+00  -6.2258125257199106D+00  -1.7152948081791512D-01
ORBIT COMPLETED.
```

ABXJRK H=350 SECS.

```
K =  5
A-ZERO IS -1.  (IAZEZ= 1)
INDICES FOR DROPPED A COEFS..   3  4  5
INDICES FOR DROPPED B COEFS..   2  3
INDICES FOR DROPPED D COEFS..   1  2  3  4  5
 THETA START =   -6.0000000000000000D-01,   THETA STEP =    4.0000000000000000D-01,   THETA STOP =    6.0000000000000000D-01
```

H=350 SECS.

```
K =  5
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


THETA =     6.00000000000000000D-01
TRUNC COEFS =    5.0848155071639798D-07   1.6013329509150756D-06   2.1816105264239839D-06,  DET =    6.1949471124163911D+31
TRUNC INDICES =          13                          14                          15
A COEFS ARE..
 -1.0000000000000000D+00   2.0000000000000000D+00  -1.0000000000000000D+00   0.
  0.                       0.
B COEFS ARE..
 -2.5301189973063973D-03   2.7807041771885522D-01   0.                        0.
  4.3816337484563047D-02   4.8121615170298022D-03
C COEFS ARE..
  3.3869865921115921D-03  -2.1928079154641655D-01   1.3979828042328042D-02   6.0957040644540645D-02
  2.2766939793594205D-02   1.0832930222134768D-03
D COEFS ARE..
  6.7583120217685833D-01   0.                       0.                       0.
  0.                       0.
STABLE
STABILITY =    0.                                                                                STABLE

        T-E                      X1-E1                    X2-E2                    U1-UDD1                   U2-UDD2
  5.9850000000000000D+05   7.6607944138872134D+06   2.3048273141722498D+06   7.6619187163819501D+06   2.3024360515319775D+06
  3.3007177400984602D-04  -1.1243024947367702D+03   2.3892626402723318D+03  -5.9641019340271774D+00  -1.7922371333383234D+00
ORBIT COMPLETED.

RUN COMPLETED.
```

ABXMO H=250 SECS.

```
K =  4
A-ZERO IS -1.  (IAZEZ= 1)
INDICES FOR DROPPED A COEFS..   3  4
INDICES FOR DROPPED C COEFS..   0  1  2  3  4
 THETA =    5.0000000000000000D-01
```

```
K =  4
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
 H=250 secs.
 THETA =      5.0000000000000000D-01
 TRUNC COEFS =    -6.2175375807069241D-08  -1.1897022063998570D-07  -1.1910624215251150D-07,  DET =      7.5542137465274648D+21
 TRUNC INDICES =              12                          13                          14
 A COEFS ARE..
   0.                        1.5000000000000000D+00  -5.0000000000000000D-01   0.
   0.
 B COEFS ARE..
  -5.4467041446208113D-04   2.2573040674603175D-01   3.5579943783068783D-02   1.0423693783068783D-02
   8.0357142857142857D-04
 D COEFS ARE..
   2.0937396600529100D-02   1.0562624007936508D-01  -1.9742683531746032D-02  -3.7355324074074074D-03
  -7.8356126543209877D-05
 E-ZERO IS   -1.0000000000000000D+00
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


 THETA =      5.0000000000000000D-01
 TRUNC COEFS =     3.8047809610389589D-07   7.4314730890251534D-07   7.6177217332480137D-07,  DET =      7.5542137465274648D+21
 TRUNC INDICES =             12                          13                          14
 A COEFS ARE..
  -1.0000000000000000D+00   2.0000000000000000D+00  -1.0000000000000000D+00   0.
   0.
 B COEFS ARE..
   1.4695767195767196D-02   3.8527336860670194D-01  -9.7874779541446208D-02  -5.3985890652557319D-02
  -4.3694885361552028D-03
 D COEFS ARE..
   2.8127865961199295D-01   3.5897707231040564D-01   9.5643738977072310D-02   1.9929453262786596D-02
   4.3209876543209877D-04
 STABLE
 STABILITY =      0.                                                                                STABLE

            T-E                       X1-F1                      X2-E2                      U1-UDD1                    U2-UDD2
   1.8750000000000000D+03  -6.6793392236013338D+05   7.9720677540623428D+06  -6.6793392839264354D+05   7.9720677459746993D+06
   1.2612069932281655D-09   6.0325191634374351D-03   8.0876435525532157D-03   5.2000117578172932D-01  -6.2064291460055597D+00

            T-E                       X1-E1                      X2-E2                      U1-UDD1                    U2-UDD2
   2.0000000000000000D+03  -1.5413521580480715D+06   7.8501104148209631D+06  -1.5413521690960727D+06   7.8501103999775339D+05
   2.3129594752765775D-09   1.1048001238296047D-02   1.4843429212073641D-02   1.1999763863164019D+00  -6.1114826960500414D+00

            T-E                       Y1-E1                      X2-F2                      U1-UDD1                    U2-UDD2
   1.8750000000000000D+03  -6.6793392236013338D+05   7.9720677540623428D+06  -6.6793392235254468D+05   7.9720677540481409D+06
   2.0127877030915897D-12  -7.5886970217204300D-06   1.4201964481239960D-05   5.2000116060889077D-01  -6.2064291355430105D+00

            T-E                       X1-E1                      X2-E2                      U1-UDD1                    U2-UDD2
   2.0000000000000000D+03  -1.5413521580480715D+06   7.8501104148209631D+06  -1.5413521580350742D+06   7.8501104148050261D+06
   2.5706123429127766D-12  -1.2997286374962734D-05   1.5936925905376460D-05   1.1999763721169262D+00  -6.1114826791326158D+00

            T-E                       X1-E1                      X2-F2                      U1-UDD1                    U2-UDD2
   5.9825000000000000D+05   7.9794676124547261D+06   5.7279719079799306D+05   7.9794592791266649D+06   5.7290929093291088D+05
   1.4051168871057941D-05   6.3333280612467378D+00  -1.1210003491781280D+02  -6.2121842629966892D+00  -4.4602246294392136D-01

ORBIT COMPLETED.
```

ABXMO H=250 SECS.

```
K =  4
A-ZERO IS -1.  (IAZE7= 1)
INDICES FOR DROPPED C COEFS..   0  1  2  3  4
INDICES FOR DROPPED D COEFS..   2
 THETA =    -6.0000000000000000D-01
```

H=250 SECS.

```
K =  4
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

THETA =    -6.0000000000000000D-01
TRUNC COFFS =     5.1322933654914429D-07    1.1707523539204734D-06    1.3830853538936478D-06,  DET =    1.3732778034411471D+24
TRUNC INDICES =              13                          14                          15
A COEFS ARE..
  0.                       3.2266366566150134D-01    4.2687120235110305D+00
  1.3140393548340332D+00                                                   -4.9054159440065651D+00
B COEFS ARE..
  4.1344667966059264D-01   1.5018719934346025D+00    2.1445679923131401D+00
 -2.5009536914548144D-02                                                   -3.1832446043505253D-02
D COEFS ARE..
  1.9706698441871665D-02   1.9319767717657933D-01    0.
 -5.7000195134854319D-01                                                   -6.0265212721572445D-01
E-ZERO IS   -1.0000000000000000D+00
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

THETA =    -6.0000000000000000D-01
TRUNC COFFS =     2.2688515338078550D-08    5.0380835865546321D-08    5.7727245566531991D-08,  DET =    1.3732778034411471D+24
TRUNC INDICES =              13                          14                          15
A COEFS ARE..
 -1.0000000000000000D+00   2.5378888019572546D+00   -2.0216179014695412D+00
  5.3959702444968087D-02                                                    4.2996939706731846D-01
B COEFS ARE..
  4.9500136559159350D-03   5.4815676102445280D-01   -7.9853717948196621D-02
 -1.0791120943856973D-03                                                   -2.3948780755732449D-02
D COEFS ARE..
  1.2127462253633290D-04   2.1601834118309214D-01    0.
 -2.1365935618491418D-02                                                   -2.3484734847065716D-01
STABLE
STABILITY =     6.2431856051012602D-01
                                                                                                    STABLE
        T-E                     X1-E1                    X2-E2                    U1-UDD1                  U2-UDD2
  5.9825000000000000D+05   7.9794576124547261D+06   5.7279719079799306D+05   7.9794598043464360D+06   5.7290222538010031D+05
  1.3165550449891836D-05   7.8081082901043491D+00  -1.0503458210724419D+02  -6.2121846302125222D+00  -4.4601695934131096D-01
ORBIT COMPLETED.
```

ABXMO H=250 SECS.

```
K =  6
A-ZERO IS -1.   (IAZEZ= 1)
INDICES FOR DROPPED A COEFS..   3  4  5  6
INDICES FOR DROPPED C COEFS..   0  1  2  3  4  5  6
INDICES FOR DROPPED D COEFS..   4  5  6
 THETA START =    2.0000000000000000D-01,   THETA STEP =    4.0000000000000000D-01,   THETA STOP =    6.0000000000000000D-01
```

H=250 SECS.

```
K =  6
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

THETA =     2.0000000000000000D-01
TRUNC COEFS =    -3.1724255726861639D-07  -1.1496097483754196D-06  -2.1241808132279108D-06,  DET =     4.7376637916643257D+29
TRUNC INDICES =              13                          14                           15
A COEFS ARE..
  0.                        1.8000000000000000D+00  -8.0000000000000000D-01   0.
  0.                        0.                       0.
B COEFS ARE..
 -1.7205654950741178D-02    7.2650890351987407D-01   7.1332102953823954D-01   1.9698778014430014D-01
  5.3120993415356573D-03   -2.6988286894002683D-04   1.1287536078178728D-05
D COEFS ARE..
  6.9811983157344773D-02   -2.6170795474481659D-01  -5.8486882766834441D-01  -1.2790076310453015D-01
  0.                        0.                       0.
E-ZERO IS   -1.0000000000000000D+00
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

THETA =     2.0000000000000000D-01
TRUNC COEFS =    -4.6776465392646783D-07  -1.7690413779053096D-06  -3.2715422407945727D-06,  DET =     4.7378637916643257D+29
TRUNC INDICES =              13                          14                           15
A COEFS ARE..
 -1.0000000000000000D+00    2.0000000000000000D+00  -1.0000000000000000D+00   0.
  0.                        0.                       0.
B COEFS ARE..
 -2.4999648416433454D-02    1.0680577305340563D+00   1.0401620996673080D+00   2.9679892509753621D-01
  8.0622908413992332D-03   -4.1099851726679212D-04   1.7223722347928337D-05
D COEFS ARE..
  1.9250179709094372D-01   -4.8194577231606654D-01  -8.6513805373869834D-01  -1.9310555396512625D-01
  0.                        0.                       0.
STABLE
STABILITY =     0.                                                                             STABLE

        T-E                      X1-E1                    X2-F2                    U1-UDD1                  U2-UDD2
  5.9825000000000000D+05   7.9794676124547261D+06   5.7279719079799306D+05   7.9794933137374510D+06   5.7245130579195138D+05
  4.3354821208137829D-05  -2.5701282724852325D+01   3.4588500614168209D+02  -6.2122080517400265D+00  -4.4566571726389196D-01

ORBIT COMPLETED.
```

H=250 SECS.

```
K =  6
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


THETA =     6.0000000000000000D-01
TRUNC COEFS =     8.6107174738437064D-09    2.8740515736452736D-08    4.8805259920765662D-08,  DET =     9.8884277205069470D+29
TRUNC INDICES =            13                          14                          15
A COEFS ARE..
  0.                          1.4000000000000000D+00    -4.0000000000000000D-01    0.
  0.                          0.                          0.
B COEFS ARE..
 -2.6430676782065671D-05     1.6243045897133730D-01    -3.0754651865881033D-02    -1.0639331425364759D-02
 -7.9739499075724566D-04     1.9841331016642960D-05    -6.4667998730288294D-07
D COEFS ARE..
  1.0241355227524469D-02     1.2776505729794979D-01     1.8827286508244842D-02     2.9364563026993583D-03
  0.                          0.                          0.
E-ZERO IS    -1.0000000000000000D+00
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX


THETA =     6.0000000000000000D-01
TRUNC COEFS =     7.0255501319107243D-07    2.4751182923907135D-06    4.4504662308435098D-06,  DET =     9.8884277205069470D+29
TRUNC INDICES =            13                          14                          15
A COEFS ARE..
 -1.0000000000000000D+00     2.0000000000000000D+00    -1.0000000000000000D+00    0.
  0.                          0.                          0.
B COEFS ARE..
  1.9734698749655694D-02     1.3035509532905366D-01    -6.7892845243105660D-01    -3.6751618366201700D-01
 -3.5080323690924181D-02     9.9932927333774036D-04    -3.5513075456009700D-05
D COEFS ARE..
  4.1680901191293101D-01     3.3948059909129706D-01     5.5471981693642233D-01     1.1946172156675629D-01
  0.                          0.                          0.
STABLE
STABILITY =     0.
                                                                                                STABLE

          T-E                       X1-E1                      X2-E2                      U1-UDD1                    U2-UDD2
 5.9825000000000000D+05     7.9794676124547261D+06     5.7279719079799306D+05     7.9794559554066424D+06     5.7295399391561326D+05
 1.9654478046299801D-05     1.1657048083762859D+01    -1.5660311762019151D+02    -6.2121819403006933D+00    -4.4605728429568607D-01
ORBIT COMPLETED.

RUN COMPLETED.
```

7.2 COEFFICIENT GENERATOR (PROGRAM ABX)

Program ABX produces multistep numerical integration methods defined by the difference equation

$$\sum_{i=0}^{k} a_i y_{n-i} + h^2 \sum_{i=0}^{k} b_i \ddot{y}_{n-i} + h^3 \sum_{i=0}^{k} c_i \dddot{y}_{n-i}$$

$$+ h^2 \sum_{i=0}^{k} d_i \ddot{y}_{n-i-\theta} + e_o y_{n-\theta} = 0 \qquad (7\text{-}1)$$

where either $a_o = -1$ and $e_o = 0$ or else $a_o = 0$ and $e_o = -1$; and θ is not an integer. Any of the coefficients (except a_o and e_o as noted) may be preset to zero. θ and d_o may also be stepped automatically through a grid of evenly spaced values to generate a sequence of methods. All remaining coefficients, including d_o if not preset or stepped through a grid, are computed using order equations defined later. The number of computed coefficients must be less than 16.

INPUT

	Item	Format
1.	K (step number)	I5
2.	IAZEZ (= 1 if $a_o = -1$, = 2 if $e_o = -1$)	I5

INPUT (Cont'd.)

	Item	Format
3.	"A" followed by the string of indices for dropped "a" coefficients. Indices for this and next three items must be in increasing order. If the first index is -1, no a's are dropped; otherwise at least one "a" is dropped.	(A2,39I2)
4.	"B" followed by the string of indices for dropped "b" coefficients. The same rules apply as for "a" coefficients above.	(A2,39I2)
5.	"C" followed by the string of indices for dropped "c" coefficients. The same rules apply as for "a" coefficients above, except that a code of -2 in place of the first index means all c's are dropped.	(A2,39I2)
6.	"D" followed by the string of indices for dropped "d" coefficients. The same rules apply as for "c" coefficients above. If d_o is input as a grid, in item 8 below, an input here of index 0 is not relevant.	(A2,39I2)

INPUT (Cont'd.)

	Item	Format
7.	Θ-start, Θ-step, Θ-stop. If the step value is blank or zero, only the start value is used and no stepping is performed for Θ.	3D25.0
8.	d_o-start, d_o-step, d_o-stop. If the step value is blank or zero, only the start value is used and no stepping is performed for d_o. If both the start value and the step value are zero or blank, there is assumed to be no grid input for d_o. In that case d_o is treated like any other coefficients "d" according to the input item 6.	3D25.0

OUTPUT

All input is printed with explanatory remarks except as follows. Θ-start is printed only if Θ-start or Θ-step is non-zero (same as non-blank). Θ-step and Θ-stop are also printed only if Θ-step is non-zero. The same rules apply to d_o-start, d_o-step and d_o-stop.

As the program loops through the grids (if any) the value of θ is printed and d_o too if d_o-start or d_o-step is non-zero, followed by the three truncation coefficients C_L, C_{L+1}, C_{L+2} defined later where L is the number of computed coefficients. The determinant of the coefficient matrix σ_{ij} defined later, as well as the coefficients for the method are also printed. Finally, a statement of stability or instability for zero step-size is printed along with the magnitude of the "largest" extraneous root. This magnitude is called the "stability" of the method.

If any of the complete sets of a's or b's or c's is preset to zero, that set is not printed. The same is true for the d's as well, provided d_o is not input as a grid.

Program Structure

ABX is divided into a main program, the main numerical subroutine T7P10 with five entry points, and eleven other subprograms: SETD, MULT, SETUM, MATINV, INLEAV, CF, RØØTER, DNEWRA, STBLTY, DFLØAT, and DGAMMA. All floating point variables and functions are in double precision.

The main driver performs the function of reading the input, setting up indices for dropped coefficients on the left hand side of the system (7-2) of order equations below, controlling the grid stepping, calling for numerical computation at five different points, and printing the output.

The subroutine T7P10 has five entry points each with the calling list

(K, NDR, IDR, DZ, TH, IAZEZ, C, ISTAB, TC, DET)

where the variables are defined as follows:

Input

K	step number.
NDR	number of dropped coefficients (may be changed as a result of calling T7P10).
IDR	string of indices for coefficients not to appear on the left hand side of the system (7-2) below. These are called "dropped" coefficients. This string may be changed as a result of calling T7P10.
DZ	value of d_o if determined by a grid for d_o. Otherwise DZ = 0.
TH	value of θ.
IAZEZ	= 1 means $a_o = -1$; = 2 means $e_o = -1$.

Output

C	string of coefficients in double precision complex form (every other value is an imaginary part = 0).
ISTAB	stability indicator (1 = stable, 0 = unstable).
TC	string of truncation coefficients C_L, C_{L+1}, C_{L+2}.

Other input is IDZRHS in named common /IDZCOM/, IDZRHS is 1 if d_o is determined via grid input, that is, if d_o-start or d_o-step is non-zero. IDZRHS is 0 otherwise. Other output is STAB in named common /STBCOM/. STAB is the value of the stability function subprogram STBLTY which is used through a call to T7P1OW.

Entry point T7P10 modifies the string IDR if necessary to insure that a_o and e_o are not computed coefficients. T7P10 also computes the constant part r_i of the vector on the right hand side of the system of order equations

$$\sum_j \sigma_{ij} \text{ (vector of computed coefficients)}_j = r_i + D_i d_o \qquad (7\text{-}2)$$

where $D_i = 0$ if d_o is not determined by grid input for d_o. Equations (7-2) are obtained by substituting $y = x^p$ for $p=0,1,\ldots,L-1$ into the difference equation (7-1) after dropping zero coefficients and putting $-a_o y_n$ and possibly $-d_o y_{n-\theta}$ on the right hand side, setting $h = 1$ and $x_o = 0$. Subroutines SETD and MULT are used for this entry point.

Entry point T7P1OX sets up the matrix σ_{ij} in (7-2) by calling subroutine SETUM and then inverts the matrix by calling MATINV.

Entry point T7P1OY sets up the vector of values D_i for the right hand side of (7-2). Subroutine SETD is used for most of the work.

Entry point T7P10Z carries out the solution of (7-2) for a particular value of DZ. DZ is zero if d_o is not determined by grid input. T7P10Z then interleaves the computed coefficients into the string C to form a complete string of coefficients using subroutine INLEAV. The truncation coefficients TC(1) = C_L, TC(2) = C_{L+1}, TC(3) = C_{L+2} are also computed at this point by reference to function subprogram CF. Subroutine MULT is used to form the products $(D_i)\ d_o$ and $(\sigma_{ij})^{-1}(r_i + D_i d_o)$.

Entry point T7P10W computes the roots of the stability polynomial at zero step-size, determines the stability or instability of the method (i.e., sets ISTAB) for zero step-size, and then computes the "stability", defined earlier, by reference to the function subprogram STBLTY.

We turn now to discuss the use of the entries in subroutine T7P10.

T7P10 should be called once any time K, NDR, IDR, or IAZEZ are set or changed.

T7P10X should be called after the first call to T7P10 and any time the matrix (σ_{ij}) should be changed. Specifically, T7P10X should be called when T7P10 has been called because of a change in K, NDR, or IDR and should also be called when Θ changes and not all d's are preset to zero or determined by grid input.

T7P10Y should be called after T7P10 is first called; after every call to T7P10 resulting from a change in K, NDR, or IDR; and when θ is set or changed; provided d_o is determined by grid input. When T7P10X and T7P10Y are both called following a call to T7P10, the order of the calls to T7P10X and to T7P10Y are not important.

T7P10Z should be called to complete the computation of coefficients after the necessary calls to T7P10, T7P10X, and possibly T7P10Y have been made and after every change in DZ (d_o grid input).

T7P10W is called to obtain the stability information for the method just obtained by calling T7P10Z. The call to T7P10W destroys the array C of coefficients, so that these coefficients should be disposed of (e.g., printed, punched, stored elsewhere, etc.) before calling T7P10W.

OTHER SUBPROGRAMS

1. SETD - Subroutine SETD is used to set up the constant matrix on the right hand side of equation (7-2). This matrix multiplies the coefficients a_i, b_i, c_i, d_i, e_o appearing on the right hand side. For program ABX, the particular coefficients on the right hand side are possibly d_o and either a_o or e_o. SETD is used in two different places. Once to get the NAI by 1

coefficient matrix for a_o and e_o. And once for the NAI by 1 coefficient matrix for d_o provided it appears on the right hand side. SETD is used by executing the statement

CALL SETD (K, NAI, NVAR, IVAR, TH, DRHS)

with calling sequence as follows:

INPUT

K step number

NAI number of order equations = number of computed coefficients a_i, b_i, e_i, d_i.

NVAR number of coefficients a_i, b_i, e_i, d_i, e_o on the right hand side of (7-2), multiplying the "coefficient matrix". NVAR is 1 for every use in the ABX package.

IVAR string of indices for the coefficients multiplying the coefficient matrix.

TH value of Θ to be used.

OUTPUT

DRHS the coefficient matrix to be generated. The functional dimension of DRHS is NAI rows by NVAR columns.

2. SETUM - Subroutine SETUM is used to set up the constant matrix σ_{ij} on the left hand side of equation (7-2), multiplying the computed coefficients. SETUM is used by executing the statement

CALL SETUM (K, NDR, IDR, TH, NAI, AM)

with the calling sequence as follows:

INPUT

K	step number.
NDR	number of dropped coefficients.
IDR	string of indices for dropped coefficients.
TH	value of Θ.
NAI	number of computed coefficients = number of order equations.

OUTPUT

AM	the matrix σ_{ij}. Functional dimensions are NAI by NAI.

3. MULT- Subroutine MULT performs a matrix multiplication $C = A * B$ where A is L rows by M columns, B is M rows by N columns, and C is L rows by N columns. MULT is used by executing the statement

CALL MULT (L, M, A, N, B, C).

4. INLEAV - Subroutine INLEAV inserts the computed coefficients G into the proper positions in a string C of coefficients which will be a_o, a_1, ..., a_k, b_o, b_1, ..., b_k, c_o, c_1, ..., c_k, d_o, d_1, ..., d_k, e_o strung out in one array. Only the odd positions are used; even positions are preset to zero. This expanded

"double-precision complex" form of the coefficients is used because the root-finding routines ROOTER and DNEWRA expect the a's in this form. INLEAV is used by executing the statement

CALL INLEAV (K, NAI, G, NDR, IDR, C)

with the calling list as follows:

INPUT

K	step number
NAI	number of computed coefficients.
G	string of computed coefficients in increasing order of indices with a's followed by b's followed by c's followed by d's.
NDR	number of coefficients dropped from the left hand side of the order equations (7-2).
IDR	indices of the coefficients dropped from the left hand side of the order equations (7-2).

OUTPUT

C	string of coefficients with the computed coefficients inserted into their proper positions. Values of the other positions of C are not changed by INLEAV.

5. CF - Function subprogram CF computes the truncation coefficient C_L in the series expansion

$$L(y;h)(x) = C_o y(x) + C_1 y'(x)h + C_2 y''(x)h^2 + \ldots + C_L y^{(L)}(x)h^L + \ldots$$

where

$$L(y;h)(x) = \sum_{i=0}^{k} \Big\{ a_i y(x-ih) + h^2 b_i \ddot{y}(x-ih) + h^3 c_i \dddot{y}(x-ih) + h^2 d_i \ddot{y}(x-ih-\Theta h) \Big\} + e_o y(x-\Theta h).$$

C_L is computed using the following formulas:

$$C_L = \frac{1}{L!} \Big\{ (R-\Theta)^L e_o + \sum_{i=0}^{k} (k-i)^L a_i + L(L-1) \Big[\sum_{i=0}^{k} (k-i)^{L-2} b_i + \sum_{i=0}^{k} (k-i-\Theta)^{L-2} d_i \Big] + L(L-1)(L-2) \sum_{i=0}^{k} (k-i)^{L-3} c_i \Big\}$$

It should be noted that this subprogram is not the same CF as used in program AB.

6. STBLTY - Function subprogram STBLTY computes the "stability" of the numerical method generated. The "stability" is the magnitude of the largest non-principle root of the characteristic polynomial $\sum_{i=0}^{k} a_i x^{k-i}$. STBLTY is used by executing a statement such as

```
STAB = STBLTY (R,K)
```

where R is the string of roots in double-precision complex form, that is, R(2i-1) is the real part of a root and R(2i) is the complex part for i = 1, ..., K. K is the number of roots.

7. Subprograms MATINV, ROOTER, DNEWRA, DFLOAT, and DGAMMA are the same as in program AB. (See Section 5 of Dyer, et.al.: Generalized Multistep Methods in Orbit Computation. System Development Corporation, document TM-4888/000/00).

EXTENSION OF PROGRAM ABX: PROGRAMS ABXMO AND ABXJRK

Program ABX has been extended to test numerical methods generated by integrating a circular orbit using the generated method if it is stable for near zero step-sizes. There are two extensions:

a) ABXJRK for "jerk" methods, where all d's except possibly d_o are zero. For each integration step there is one "prediction" using the analytic solution and then one correction using the generated method.

b) ABXMO for "multi-offgrid" methods, where all c's are zero. For each integration step, there is an off-grid "prediction" for $y_{n-\theta}$ using the analytic solution, an off-grid correction of $y_{n-\theta}$ using one of a pair of generated correctors, and then an on-grid

"prediction" for y_n using the analytic solution, and finally an on-grid correction of y_n using the second of a pair of generated correctors. The off-grid corrector has the same preset coefficients as the on-grid corrector except that $a_o = 0$, $e_o = -1$ for the off-grid corrector and $a_o = -1$, $e_o = 0$ for the on-grid corrector.

Input

The input for ABXJRK and ABXMΘ are the same as for ABX except that a card must be inserted in front of all other input, giving the step-size in format D25.0. The input values of IAZEZ are ignored in both ABXJRK and ABXMΘ. The values of IAZEZ are set in the main programs according to the program needs.

Output

a) ABXJRK: Each case is printed on a separate page in the same format as ABX. In addition, for each stable method the result of integration is printed in the following format:

T-E	X1-E1	X2-E2
time of last integration step (seconds)	first coordinate of true colution (=X1)	second coordinate of true solution (=X2)
relative error $\frac{\sqrt{E1^2 + E2^2}}{\sqrt{X1^2 + X2^2}}$	signed error in first coordinate E1 = X1-U1	signed error in second coordinate E2 = X2-U2

(continued)

U1-UDD1	U2-UDD2
first component of integrated solution (=U1)	second component of integrated solution (=U2)
first component of acceleration evaluated at (U1,U2)	second component of acceleration evaluated at (U1,U2)

The message "ORBIT COMPLETED." is printed if the orbit has been completed. Otherwise, the message "ORBIT INCOMPLETE." is printed. This latter case occurs when the relative error gets too large, i.e., when $\frac{(U1)^2 + (U2)^2 - A^2}{A^2}$ is larger than $\frac{1}{10}$ where A is the radius of the true orbit ($A = 8 \times 10^6$ meters).

b) ABXMO: For each case, the pair of correctors is printed on a page with the result of integration printed as for ABXJRK.

Subprograms Used.

Subroutine TEST is the main subprogram performing the task of integration. TEST has no formal parameters. It computes essential parameters for the true solution, calls subroutine START to start the integration tables, then iteratively moves points back in the tables and calls subroutine STEP to integrate

one step at a time until the orbit is completed or the error is too large, then calls OUTPUT for the summary of integration, and finally prints the message "ORBIT COMPLETED." or "ORBIT INCOMPLETE."

Subroutines START and STEP also use function subprogram FUNC2 to compute components of the acceleration. In ABXJRK, subroutines START and STEP also use function subprogram FUNC3 to compute components of the third derivative $U^{(3)}$ of motion. The formula used is

$$U^{(3)} = - \frac{\mu V}{|U|^3} \tag{7-3}$$

where V is the velocity vector determined from the true solution and U is the vector position (the true value for starting, and integrated value after starting). Formula (7-3) is obtained by assuming the magnitude of $|U|$ of the motion to be constant.

8. Equivalent Methods

The multistep algorithm written in the matrix notation [8] is

$$\bar{z}_{n+1} = A\bar{z}_n - \ell F(z_n)$$

where $\bar{z}_n$ is a vector (basis) of saved values. In the above equation, matrix A depends on predictor and corrector coefficients and $F = hP - hf(x, y^P_{n+1})$ where P is the predicted derivative estimate at time $n+1$. Integration with respect to another basis, another vector, $\bar{y}_n$, of saved values, where $\bar{y}_n = Q\bar{z}_n$ is defined by the algorithm

$$\bar{y}_{n+1} = QAQ^{-1}\,\bar{y}_n - Q\bar{\ell}F(Q^{-1}\,\bar{y}_n) \quad .$$

It is to be noted that the transformed method will give identical answers, if transformed starting values are used to begin the integration and if the inverse mapping $Q^{-1}\,\bar{y}_n$ is performed after computation. Thus, the equivalence has a rather trivial meaning.

An example will serve to illustrate different results obtainable in one step, if the Adams first-order method and an equivalent method are applied to exact starting values.

Given the differential equation

$$y = f(x,y) \equiv \frac{4y}{x}$$

with $y(0) = 0$ and $f(0,0) = 0$, whose solution is $y = x^4$ with tabulated values

$x =$	0	1	2	3
$y =$	0	1	16	81
$y' =$	0	4	32	108 .

The Adams 4-value method, as given by Gear [9] is defined by

$$A = \begin{pmatrix} 1 & 23/12 & -4/3 & 5/12 \\ 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \ell = \begin{pmatrix} 3/8 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\bar{z}_n = (y_n, y'_n, y'_{n-1}, y'_{n-2}) .$$

Straightforward application of this method, using starting information at $x = 0,1,2$ yields approximately the result at $x = 3$

7.

$$x = 3.0$$
$$y = 76.5$$
$$y' = 95.0 .$$

The equivalent 4-valued modified method is defined by

$$QAQ^{-1} = B = \begin{pmatrix} -4 & 5 & 4 & 2 \\ 1 & 0 & 0 & 0 \\ -12 & 12 & 8 & 5 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad Q\bar{\ell} = \begin{pmatrix} 3/8 \\ -1/24 \\ 1 \\ 0 \end{pmatrix}$$

$$Q\bar{z}_n = (y_n, y_{n-1}, y'_n, y'_{n-1}) \quad .$$

The basis change matrix Q is given in [9]. Application of the method using starting information at $x = 1,2$, gives the result

$$x = 3.0$$
$$y = 78.7$$
$$y' = 102.7$$

In neither method is the derivative reevaluated -- i.e., algorithm PEC is used. It is important to note the difference in equivalent solutions and the fact that in the present case, the transformed method gives a much better answer.

The above 4-value method (the Q transform) is given by Gear [9], where the error is stated to be $O(h^5)$. It is seen, however, that the method is not exact for a fourth degree polynomial.

8.1 Linear Transformations of Adams Class II Methods

The Adams Class II, 12-value method makes use of the basis (type 2-10)

$$\bar{y}_n = (y_n, y_{n-1}, y''_n, y''_{n-1}, \ldots, y''_{n-9})$$

the matrix A and the vector $\bar{\ell}$ are readily computed from the usual Adams predicting and correcting equations. The defining formulas for the entries of A and the components of $\bar{\ell}$ are given in [9]. The 12-value Adams method uses the Cowell corrector of order 11 and Störmer predictor, order 10.

Two different basis were considered in this analysis.

(1) The 4-8 basis

$$\bar{y}_n = (y_n, y_{n-1}, \ldots, y_{n-3}, y''_n, y''_{n-1}, \ldots, y''_{n-7}) \ .$$

(2) The 6-6 basis

$$\bar{y}_n = (y_n, y_{n-1}, \ldots, y_{n-5}, y''_n, y''_{n-1}, \ldots, y''_{n-5}) \ .$$

A program was written to compute the necessary change of basis transformations and to apply them to the above matrix A and vector $\bar{\ell}$. The coding was validated by comparing the 6-6 transformation with SDC Report [2].

Computations were performed on a 2-hr. circular orbit for 84 revolutions. Relative error in the basic Adams algorithm described above was $.13 \times 10^{-9}$. Simple transformation to the 4-8 basis yielded a relative error of $.21 \times 10^{-2}$, while transformation to the 6-6 basis yielded a relative error of $.79 \times 10^{-3}$. All computations were done with one correction and no final reevaluation of the derivative.

In addition, variant Adams type methods were transformed. Instead of using the usual Adams $\bar{\ell}$ vector, $\bar{\ell} = (\beta_0,0,1,0,\ldots,0)^T$, as above, a class of stable methods was formed by taking the first two components in the $\bar{\ell}$ vector, $\bar{\ell} = (\beta_0',\beta_1',0,\ldots,0)^T$, as parameters and allowing them to vary over the interval $[-1,1]$. It is easy to show stability for sufficiently small h and any β_0', β_1'. The motivation for this is explained. Straightforward transformation of the Adams' method into the 6-6 basis, for example, yields a vector $Q\bar{\ell}$ with approximate components

$$\begin{pmatrix} -.060 \\ 0 \\ -.002 \\ -.017 \\ -.018 \\ -.018 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

These, successively, are the coefficients used in the correction of back-function values. One expects these coefficients to decrease significantly in magnitude, if a good method results. The Class I transforms demonstrate this property, as well as alternating in sign. Rather than implement a rigorous program of optimization of truncation error in the transformed methods, it was decided to examine the components of $Q\bar{\ell}$, where the first two components of $\bar{\ell}$ vary in the abovementioned region.

The procedure described in the preceeding paragraph was carried out for both basis change transformations Q_{4-8}, Q_{6-6}. Parameters β_0' and β_1' were permitted independently to assume values in the set -1 (.2) 1. Sub-regions were then examined with a finer mesh. The best method obtained in this manner was a Q_{4-8} transformation of the Adams

variant method, defined by $\beta_0' = .163$, $\beta_1' = .15$, yielding a vector

$$\begin{pmatrix} .163 \\ .150 \\ .740 \times 10^{-1} \\ .740 \times 10^{-4} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Selection of parameters has a large effect on propagated error. However, no selection of parameters obtained by transforming variant Adams methods to either basis, yielded a method comparable in efficiency to the untransformed Adams method. The relative error in the best transformed method was of the order 10^{-5} on the same arc, as discussed above. One difficulty, perhaps, arises from the fact that this type of new method makes explicit use of a derivative prediction. As fewer derivative values are used in the prediction (because basis change has reduced the number of time points involved), derivative prediction has approached, in a partial sense, explicit numerical differentiation.

Analytical determination of the parameters for increased order or numerical determination by minimization of propagated error on particular orbit types can, perhaps, produce superior methods, but in

view of R. Danchick's results (given immediately below), the expected gain in efficiency over the Adams method is not great. Danchick's method (Nordsieck basis) has a great advantage in simplicity of step-size change, if this is required.

In Appendix B (partially supported under Contract NAS5-11940) it is shown that properly maximal polynomial degree (p.d.) Nordsieck, Gear Class I and II methods were not equivalent in the sense of Gear and Descloux to the classical Adams-Moulton and Cowell methods and, indeed, that this sense of equivalence must necessarily be abandoned as logically irrelevant. It was further shown that maximal polynomial degree could not be maintained after the first step in the presence of non-trivial stabilizing parameters, unless a special order-preserving mechanism was introduced.

A Class II integrator was developed by R. Danchick, using a Nordsieck-type basis. The method is of maximal order $p = 13$ and coefficients are altered at every step for the first p steps, in order to preserve this order. This method (N-D) was tested against the Cowell method of the same order on the 8×10^6 meter radius, circular trajectory (2-dimensional problem). A comparison of the error after 84 revolutions is given below. The length of the error vector in meters, is given at the indicated step-sizes.

Step-size (secs.)	180	240	300
N-D Method	.48	13.2	103.6
Cowell Method	.15	4.6	45.0

The performance of the N-D method is apparently superior to that of the Nordsieck method, implemented at GSFC some time ago. As in the case of the Cowell method, the new methods can probably be optimized with respect to the placement of the $h = 0$ extraneous roots. An important advantage of the N-D method is the ease of step-size change. Neglecting that advantage, the performance on orbit problems can probably be expected to be comparable to that of the Cowell method. However, since varistep procedures are still used at GSFC, the N-D methods should be considered for implementation and further tests should be made.

9. Iterative Starting and Class II Interpolators

9.1 Iterative Starting

Assume k independent difference equations of the form

$$\sum_{i=0}^{k} a_{i,j}\, y_{n-i} + h \sum_{i=0}^{k} b_{i,j}\, \dot{y}_{n-i}, \qquad j = 1,2,\ldots,k \quad .$$

Let $h > 0$ and let y_{n-k}, $\dot{y}_{n-k}$ be initial values for the equation $\dot{y} = f(t,y)$. We wish to generate starting values for a multistep integration. Assume an approximate starting solution to be given.

We begin by solving Eq. j for y_{n-j} and delay substitution of these new values into the equations until one sweep is completed. After the $m+1^{th}$ iteration, we may write the difference equations

$$\begin{aligned}
&a_{0,1}(y_{n,m+1} - y_{n,m}) + a_{1,1}(y_{n-1,m} - y_{n-1,m-1}) \\
&\quad + \cdots + a_{k-1,1}(y_{n-k+1,m} - y_{n-k+1,m-1}) \\
&\quad + \cdots + h\{b_{0-1}(\dot{y}_{n,m} - \dot{y}_{n,m-1}) + \cdots + b_{k-1,1}(\dot{y}_{n-k+1,m} - \dot{y}_{n-k+1,m-1}\} = 0 \\
&a_{0,2}(y_{n,m} - y_{n,m-1}) + a_{1,2}(y_{n-1,m+1} - y_{n-1,m}) \\
&\quad + \cdots + a_{k-1,2}(y_{n-k+1,m} - y_{n-k+1,m-1}) \\
&\quad + \cdots + h\{b_{0,2}(\dot{y}_{n,m} - \dot{y}_{n,m-1}) + \cdots + b_{k-1,2}(\dot{y}_{n-k+1,m} - \dot{y}_{n-k+1,m-1}\} = 0 \, .
\end{aligned}$$

- - - - - - - - - - - - - - - -

$$a_{0,k-1}(y_{n,m} - y_{n,m-1}) + a_{1,k-1}(y_{n-1,m} - y_{n-1,m-1})$$
$$+ \cdots + a_{k-1,k-1}(y_{n-k+1,m+1} - y_{n-k+1,m})$$
$$+ \cdots + h\{b_{0,k}(\dot{y}_{n,m} - \dot{y}_{n,m-1}) + \cdots + b_{k-1,k-1}(\dot{y}_{n-k+1,m} - \dot{y}_{n-k+1,m-1})\} = 0 .$$

The iteration can be written

$$(L + U)\, \nabla_m + D\, \nabla_{m+1} + h\, B\, J_m\, \nabla_m = 0$$

where $A = (a_{i,j})$ is partitioned into upper and lower triangular matrices U, L and a diagonal matrix D, $B = (b_{i,j})$,

$$\nabla_j = (\nabla_{0,j}, \nabla_{1,j}, \ldots, \nabla_{n-1,j})^T \quad \text{and} \quad \nabla_{i,j} = y_{n-i,j} - y_{n-i,j-1} .$$

and

$$J_m = \begin{pmatrix} \frac{\partial f}{\partial y}(t_n, \xi_0) & & \\ & \frac{\partial f}{\partial y}(t_{n-1}, \xi_1) & \\ & & \frac{\partial f}{\partial y}(t_{n-k+1}, \xi_{k-1}) \end{pmatrix}$$

where ξ_i is on the open interval (y_m, y_{m-1}).

If the differential equation is linear, J_m is constant and the iteration

$$\nabla_{m+1} = - D^{-1}(L + U + h\ B\ J)\nabla_m \tag{9-1}$$

takes the well-known form for linear algebraic equations. This (Jacobi) iteration is convergent if the spectral radius of $D^{-1}(L + U + h\ B\ J)$ is less than 1. This condition is, in turn, satisfied for sufficiently small h if the spectral radius of $D^{-1}(L + U)$ is less than 1. It is easy to prove convergence from the fact that $\nabla_m \rightarrow 0$.

If new values of the function are substituted into the equations as soon as they are computed (and if the equation is still linear) an iteration (Gauss-Seidel) results of the form

$$\nabla_{m+1} = - (L + D)^{-1} (U + h\ B\ J)\nabla_m \ . \tag{9-2}$$

It is well known [12] that if A is non-negative, this iteration converges if the above iteration (9-1) converges, and if it is convergent, it converges faster. $L + D$ is assumed invertable.

There is a further iteration corresponding to the case in which new values of the derivative are substituted as soon as possible. The form of this iteration involves a decomposition also of the matrix B and is clear.

$$\nabla_{m+1} = - (D + L + h(D' + L')J)^{-1} (U + h\ U'\ J)\nabla_m \ .$$

The primed matrices refer to the obvious decomposition of B.

It is well known [12] that the Jacobi and Gauss-Seidel iterations converge for any initial approximation (for sufficiently small h) if A is non-negative and (strictly or irreducibly) diagonally dominant.

Both the Jacobi and Gauss-Seidel iterations converge if A is symmetric-positive definite.

There are various other iterations possible. One of these (Kaczmarz), applied simply to the linear algebraic $n \times n$ system, $Ax = b$, consists in regarding the individual equations as hyperplanes of dimension $n-1$ and projecting an arbitrary point in n space, cyclically on each of the hyperplanes in any fixed order. This method is universally convergent (although, normally, slowly convergent) and can be adapted to the system of difference equations.

Clearly, a further iteration can be defined for solution of the difference equations by writing for invertable A

$$\nabla_{m+1} = hA^{-1} BJ \nabla_m \quad . \tag{9-3}$$

This iteration will converge if h is sufficiently small, since convergence depends on the spectral radius of the matrix $hA^{-1} BJ$.

In the case where the differential equation is non-linear, J_m is not constant. All the above iterations remain convergent if the eigenvalues of the matrix in question are less than unity in magnitude

at the solution and the initial approximation is sufficiently good.

Independent Difference Equations

It is necessary, for the starter problem, to have vectors

$$a_{0,j}, a_{1,j}, \ldots, a_{k-1,j}$$

independent, in order to obtain a unique solution at $h = 0$. There is an upper limit to the order of the difference equations to be solved if the equations are to be independent.

There is a unique difference equation

$$\sum_{i=0}^{k} a_i \, y_{n-i} + h \sum_{i=0}^{k} b_i \, y_{n-i} = 0$$

of maximal order m $(a_0 = -1)$. By arbitrarily predetermining any one coefficient, other methods of order $m-1$ result. Let these new coefficients be called $\bar{a}_i$, $\bar{b}_i$. Assume, for simplicity, $\bar{a}_1$ has been predetermined. We have new coefficients

$$\bar{a}_0 = -1, \bar{a}_1, \bar{a}_2(\bar{a}_1), \ldots, \bar{a}_k(\bar{a}_1), b_0(\bar{a}_1), \ldots, \bar{b}_k(\bar{a}_1) \quad .$$

If any other coefficient, say $\bar{a}_2$, had been predetermined to the value $\bar{a}_2 = \bar{a}_2(\bar{a}_1)$, we would, in general, have $\bar{a}_1(\bar{a}_2) = \bar{a}_1$ and the

identical set of coefficients results. Thus, any difference equation obtained by predetermining any coefficient, can be obtained by predetermining a single coefficient. These difference equations can be obtained from the system, e.g.

$$D \begin{pmatrix} a_1 \\ a_2 \\ a_{j-1} \\ a_{j+1} \\ \vdots \\ a_k \\ b_0 \\ b_1 \\ \vdots \\ b_k \end{pmatrix} = \begin{pmatrix} 1 \\ k^2 \\ k^3 \\ \vdots \\ k^{2k} \end{pmatrix} - a_j \begin{pmatrix} 1 \\ j^2 \\ j^3 \\ \vdots \\ j^{2k} \end{pmatrix} .$$

The matrix D is constant and the coefficient a_j varies. Clearly, only two independent solution vectors can be obtained. Therefore, there are two independent methods of order $m-1$. In general, these are k independent equations of order $m-k+1$.

Additional off-grid points may be introduced to obtain high-order. If these are chosen as midpoints, as seems natural, essentially the same result can be obtained (as far as order is concerned) by taking a low-order method, such as the Adams method, using half the step-size and retaining every other point for starting.

The same results apply to Class II methods, or with general Class I/II methods.

9.2 Hermite Type Interpolating Polynomials

The GSFC trajectory determination systems, GEOSTAR and GTDS, use Hermite interpolation equations (Class I) for computing position at non-step points, as required by the observations or the integration process. Because position is integrated with a Class II operator, it is consistent and, perhaps, more accurate to interpolate position with the quasi-Hermite (Class II) method.

The coefficient generator, GENCOB, unlike other coefficient generators, constructs the quasi-Hermite weighting polynomials H_i, $\bar{H}_i$ in the form

$$P(t_n - ht) = \sum_{i=0}^{k} H_{i,0}(k-t)\, y_{n-i} + h^2 \sum_{i=0}^{k} H_{i,2}(k-t)\, \ddot{y}_{n-i} \ .$$

Other coefficient generators simply evaluate the $H_{i,0}$, $H_{i,2}$ at certain values of t. Therefore, GENCOB is capable of deriving continuous interpolators for the given problem.

The polynomials $H_{i,0}$, $H_{i,2}$ are given below for $k = 5,6$ (Class II). These interpolators are of order 10 and 11, respectively. For even k, the quasi-Hermite interpolator is not unique and can-

not be obtained by GENCOB without eliminating a further condition (setting some $H_{i,0}$ or $H_{i,2}$ identically equal to zero). Therefore, for $k = 6$, we have set $H_{3,2} \equiv 0$.

The polynomials $H_{i,0}$ and $H_{i,2}$ are given below, as well as the primary error coefficient for values of t: $.5h, 1.5h, \ldots, (k-.5)h$. In this way, the position error at mid-point interpolation can be estimated. Interpolation is, of course, exact at step-points. The polynomial P ; as well as the $H_{i,0}$, $H_{i,2}$, are uniquely determined by the conditions

$$\left.\begin{aligned} P(t_n - t) &= y(t_n - t) \\ \ddot{P}(t_n - t) &= \ddot{y}(t_n - t) \end{aligned}\right\} t = 0, h, \ldots, kh$$

with the condition $\ddot{P}(t_n - 3h) = \ddot{y}_{n-3}$ eliminated for $k = 6$.

Computations were performed on the CDC 6600 and coefficients are exact, to at least 16 significant digits. Alteration of GENCOB was done to print the weighting polynomials and the error. The $H_{i,p}$ are

$$H_{i,p} = \sum_{j=0}^{2k+1} a_{i,j,p}\, t^j \qquad (k = 5)$$

$$H_{i,p} = \sum_{j=0}^{2k} a_{i,j,p}\, t^j \qquad (k = 6)$$

and coefficients are listed in the order $a_{i,0}, a_{i,1}, \ldots, a_{i,m}$. The $H_{i,p}$ are printed in the order $H_{k,0}, H_{k-1,0}, \ldots, H_{0,0}, H_{k,2}, H_{k-1,2}, \ldots, H_{0,2}$.

H MATRIX

```
 1.000000000000000D+00 -2.685404679708477D+00  0.                     1.026898734177215D+01 -1.993710443037975D+01
 1.850010548523207D+01 -9.036511075949367D+00  3.292787447257384D+00 -6.835443037974684D-01  8.662974683544304D-02
-6.131329113924051D-03  1.857978519370924D-04

 0.                     7.767548906789413D+00  0.                    -4.382911392405063D+00 -2.602202004219409D+01
 4.574808880801687D+01 -3.159322257383966D+01  1.171446466244726D+01 -2.528085443037975D+00  3.167853375527426D-01
-2.122890295358650D-02  5.913674721902570D-04

 0.                    -5.178365937859609D+00  0.                    -2.155063291139241D+01  9.318222573839662D+01
-1.226888185654008D+02  8.895319092827004D+01 -3.101595464135021D+01  7.240506329113924D+00 -1.019251054852321D+00
 7.977320675105485D-02 -2.673091676562330D-03

 0.                    -3.260452627641235D+00  0.                     1.484177215189873D+01 -2.915427215189873D+01
 3.492576476793249D+01 -2.742958860759494D+01  1.344870780590717D+01 -4.013449367088608D+00  7.060917721518987D-01
-6.724683544303797D-02  2.673091676562330D-03

 0.                     2.012830840046030D+00  0.                     1.155063291139241D+00 -1.746453059071730D+01
 2.272204641350211D+01 -1.194494969451477D+01  2.828652426160338D+00 -1.645569620253165D-01 -5.472046413502110D-02
 1.074630801687764D-02 -5.913674721902570D-04

 0.                     3.438434962738780D-01  0.                    -3.322784810126582D-01 -6.042985232067511D-01
 7.928138185654008D-01 -4.918776371308020D-02 -2.686577004219409D-01  1.491297468354430D-01 -3.553533755274262D-02
 4.087552742616034D-03 -1.857978519370924D-04

 0.                    -1.117869472299852D-01  5.000000000000000D-01 -1.082032348804501D+00  1.379483122362869D+00
-1.081443389592124D+00  5.351661392405063D-01 -1.697878993369500D-01  3.441455696202532D-02 -4.307313642756681D-03
 3.032700421940928D-04 -9.190001278608874D-06

 0.                     6.164721354594772D-01  0.                    -9.731012658227848D+00  2.332388361462729D+01
-2.411631944444444D+01  1.372882383066245D+01 -4.693650165762508D+00  9.890559071729958D-01 -1.258570675105485D-01
 8.869549929676512D-03 -2.657109065336913D-04

 0.                    -2.059066250205491D+00  0.                    -1.409282700421941D+01  5.624402250351617D+01
-6.991939170182841D+01  4.301073312236287D+01 -1.506285789632309D+01  3.137921940928270D+00 -3.829113924050633D-01
 2.504395218002813D-02 -6.672740058816008D-04

 0.                    -3.370047673045142D+00  0.                    -2.049929676511955D+00  2.231513713080169D+01
-2.881175281293952D+01  1.542832278481013D+01 -3.867785563592526D+00  3.188291139240506D-01  4.821554149085795D-02
-1.165611814345992D-02  6.672740058816008D-04

 0.                    -6.164721354594772D-01  0.                     3.560126582278481D-01  1.832366385372714D+00
-2.399305555555556D+00  7.093011603375527D-01  1.624001657625075D-01 -1.765559071729958D-01  4.773206751054852D-02
-5.744549929676512D-03  2.657109065336913D-04

 0.                    -1.479533125102745D-02  0.                     1.751054852320675D-02  1.586673699015471D-02
-2.067950070323488D-02 -7.885021097046414D-03  1.617503013863773D-02 -7.779535864978903D-03  1.780063291139241D-03
-2.021800281293952D-04  9.190001278608874D-06
```

1.000000000000000D+00	-2.760540784579330D+00	0.	1.057635818964276D+01	-2.010121738261266D+01
1.879915150288217D+01	-1.052323521738223D+01	3.774267483619084D+00	-8.892772210734337D-01	1.370946085504233D-01
-1.332450672223509D-02	7.413399254397287D-04	-1.801224998531063D-05		
0.	6.150514348690202D+00	0.	-3.686072989472198D+00	-2.068926908257848D+01
3.921666000108576D+01	-2.982768040064877D+01	1.262948741415229D+01	-3.277650560132184D+00	5.346349442026072D-01
-5.356680277994631D-02	3.016257332686362D-03	-7.312985196174131D-05		
0.	-5.141425954621473D+00	0.	-2.683120695019621D+01	1.098984947874261D+02
-1.453883558992305D+02	9.857408636725759D+01	-3.971773492290306D+01	1.008521161566508D+01	-1.634247022431838D+00
1.643010849876342D-01	-9.353884349133530D-03	2.307783957527928D-04		
0.	2.500087319226223D-01	0.	2.339450172789940D+01	-7.292408626472316D+01
9.278960355845762D+01	-6.502901554908610D+01	2.805926279287941D+01	-7.792263871450576D+00	1.399454770581244D+00
-1.572404370468549D-01	1.005381315401334D-02	-2.792725876114818D-04		
0.	2.949770823284165D+00	0.	-4.125510747664563D+00	-7.482615416877809D-01
2.250907967245878D-02	5.675945544472781D+00	-5.873127855392508D+00	2.680148324525843D+00	-6.697164317145384D-01
9.527418625345696D-02	-7.262160145067554D-03	2.307783957527928D-04		
0.	-1.273191059825333D+00	0.	3.645599219202067D-01	4.728452436408857D+00
-5.738614260517615D+00	1.716623396819588D+00	6.463650512830877D-01	-6.004353702587667D-01	1.823142690971228D-01
-2.825034708374378D-02	2.249092008559012D-03	-7.312985196174131D-05		
0.	-1.751361048708529D-01	0.	3.073708478706045D-01	-1.641129522329131D-01
2.990460176501030D-01	-5.867241414328625D-01	4.814800363616998D-01	-2.057329172759653D-01	5.046486171498022D-02
-7.193177608311040D-03	5.555420735026363D-04	-1.801224998531063D-05		
0.	-1.042039844600405D-01	5.000000000000000D-01	-1.097198812214968D+00	1.392531498544443D+00
-1.102455382415666D+00	5.668030513766625D-01	-1.936985540572312D-01	4.432022854268761D-02	-6.707612979773578D-03
6.442887550203513D-04	-3.558183657658901D-05	8.607454427023493D-07		
0.	9.350671326835377D-01	0.	-1.015800202813301D+01	2.319974294057506D+01
-2.408675515526855D+01	1.435280400980746D+01	-5.351609725239945D+00	1.292330215705365D+00	-2.023459220548447D-01
1.985151938595026D-02	-1.110002804080145D-03	2.701534306038875D-05		
0.	-9.665564506238740D-01	0.	-1.481219461104131D+01	4.922755520808980D+01
-6.126690265899512D+01	4.036800400305908D+01	-1.596069599018291D+01	3.986732236431948D+00	-6.352075750057028D-01
6.267406471102781D-02	-3.492275985411791D-03	8.404954247789292D-05		
0.	1.376037664122140D+00	0.	-3.633549485940488D-01	-5.184100910053656D+00
6.253183487277730D+00	-1.843427958966239D+00	-7.354379280973202D-01	6.722543883306819D-01	-2.045641150900910D-01
3.188556260132317D-02	-2.559291072996499D-03	8.404954247789292D-05		
0.	3.037396659730330D-01	0.	-4.094788213819549D-01	-1.082739370620702D-01
8.884788472655221D-03	6.160951490479660D-01	-6.417845293387996D-01	2.954947726673905D-01	-7.470879125315690D-02
1.077978942814435D-02	-8.351018962678449D-04	2.701534306038875D-05		
0.	7.582962769944705D-03	0.	-1.516646341046737D-02	1.304837618157379D-02
-2.101199282354248D-02	3.163691213615622D-02	-2.391065472028124D-02	9.905671580662290D-03	-2.400299337016898D-03
3.410187128262585D-04	-2.639183529798014D-05	8.607454427023493D-07		

Error Coefficients Are:

k = 5 (Class II)

t	$C_{12} \times 10^6$
.5h	-3.3
1.5h	3.2
2.5h	-3.2
3.5h	3.2
4.5h	-3.3

k = 6 (Class II)

6	$C_{13} \times 10^6$
.5h	-1.7
1.5h	1.6
2.5h	-1.6
3.5h	1.6
4.5h	-1.6
5.5h	1.7

10. References

[1] Baker, R.M.L., Jr. and M.W. Makemson, An Introduction To Astrodynamics. Academic Press, New York, 1960.

[2] Chesler, L. and S. Pierce. The Application of Generalized, Cyclic and Modified Numerical Integration Algorithms To Problems of Satellite Orbit Computation. System Development Corporation report TM-4717 1971.

[3] Conte, S.D. "The Computation of Satellite Orbit Trajectories" in F.L. Alt and M. Rubinoff (Eds.), Advances in Computers, Vol. III. Academic Press, New York, 1962.

[4] Dallas, S.S. and E.A. Rinderle. A Comparison of Cowell's Method and a Variation-of-Parameters Method for the Computation of Precision Satellite Orbits. J.P.L. report TM-392-101, 1972.

[5] Deutsch, R. Orbital Dynamics of Space Vehicles. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.

[6] Dyer, J. et al. Generalized Multistep Methods in Orbit Computation: Studies in Existence Theory, Efficiency, Optimization. System Development Corporation report TM-4888, 1972.

[7] Forsythe, G.E. and W.R. Wasow. Finite Difference Methods for Partial Differential Equations. John Worley, New York, 1960.

[8] Gear, C.W. Numerical Initial Value Problems in Ordinary Differential Equations. Prentice-Hall, Inc., Englewood Cliffs, N.J.1971.

[9] Gear, C.W. The Numerical Integration of Ordinary Differential Equations of Various Orders. Argonne National Laboratory report ANL-7126, 1966.

[10] Henrici, P. Discrete Variable Methods in Ordinary Differential Equations. John Wiley, New York, 1962.

[11] Maury, J.L., Jr. and G.P. Brodsky. Cowell Type Numerical Integration as Applied to Satellite Orbit Computation. Goddard Space Flight Center (NASA) document X-553-69-46, 1969.

[12] Varga, R.S. Matrix Iterative Analysis. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.

ERROR PROPAGATION IN SECOND-ORDER PREDICTOR-CORRECTOR METHODS AND APPLICATIONS TO ORBIT COMPUTATION

by

J. Dyer, Pan Science Systems, Los Angeles, California

C. E. Velez, Mission Support Computing and Analysis Division, Goddard Space Flight Center (NASA), Greenbelt, Maryland

ABSTRACT

The problem of efficiency in the numerical solution of $\ddot{y} = f(t,y)$ is considered. Some basic results applicable to the optimization of first-order (Class I) corrector methods are extended, an accumulated error estimate derived for second-order (Class II) methods and families of such methods are constructed, containing the Cowell method, for the purpose of improving the efficiency of the latter. A stability analysis is given for selected methods combined with the Störmer predictor. Results of computations of a circular orbit for one week are given, verifying the error reduction obtainable with this type of analysis.

APPENDIX A

ERROR PROPAGATION IN SECOND-ORDER PREDICTOR-CORRECTOR METHODS AND APPLICATIONS TO ORBIT COMPUTATION

1. INTRODUCTION

Hull and Newbery [2], [3], derived estimates for the error propagated by numerical integration algorithms in the solution of a first order differential equation and constructed procedures for the minimizing of such error. Much study has been done on the efficiency of numerical integration methods for first-order equations.

Second-order differential equations are important in many applied problems where the question of efficiency is paramount because of the amount of computer machine time consumed in obtaining accurate solutions. Improvement of the efficiency of orbit computation algorithms, in part, motivated the present study.

A propagated error estimate for (corrector) methods of solving $\ddot{y} = f(t,y)$ is derived. On the basis of this estimate, one-parameter (r) families of methods are derived and constructed for the purpose of improving the error propagation in the Adams Class II or Cowell method. The families contain the Cowell method for $r = 0$.

Theorems of importance in improvement of efficiency of Class I methods are extended to the second-order equations. A stability analy-

sis is done of selected methods and computations of a circular orbit performed for the purpose of comparison with the Cowell method.

It is found that simple optimization procedures can effectively reduce propagated error of the Cowell method without undue loss of stability. For certain algorithms (PEC) the interval of stability can be extended over that of the Cowell method (for the circular orbit problem) by the procedures which reduce propagated error.

Methods constructed are closely related to the Cowell method. The non-principal roots of the characteristic polynomial of the iterated corrector are not taken at the origin. The methods have been derived and tested in unsummed form, [1, p. 237], but the derivation and implementation of corresponding summed forms (to yield variant Gauss-Jackson formulas) is not difficult. Additional per-step machine time required (over the Cowell or Gauss-Jackson method) should be negligible in situations where efficiency is governed by the number of function evaluations. Finally, the results of Hull and Creemer [4] concerning optimum predictor/corrector Class I algorithms are shown to apply to Class II methods.

2. AN ESTIMATE FOR PROPAGATED ERROR

An estimate for the error propagated in the numerical solution of the special second-order equation $\ddot{y} = f(y,t)$ by the Class II corrector formula

$$\sum_{i=0}^{k} a_i\, y_{n-i} + h^2 \sum_{i=0}^{k} b_i\, \ddot{y}_{n-i} = 0, \qquad a_0 = -1 \;, \qquad (2.1)$$

can be derived in a manner extending that given by Hull and Newbery [2]. Throughout the remainder of the paper, Σ is written for $\Sigma_{i=0}^{k}$ and Σ^{k-1} is written for $\Sigma_{i=0}^{k-1}$.

Principle Roots of the Characteristic Polynomial

We assume that the coefficients in method (2.1) satisfy at least the first three order equations of the type

$$\sum a_i = 0$$

$$\sum i\, a_{k-i} = 0$$

$$\sum i^2 a_{k-i} + 2 \sum b_i = 0 \qquad (2.2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum i^j a_{k-i} + j(j-1) \sum i^{j-2} b_{k-i} = 0 \;.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The order of the difference equation (2.1) (i.e., of the associated difference operator) is defined to be p if the coefficients sat-

isfy the first $p+2$ order equations, but fail to satisfy the succeeding one. The polynomial degree is $p-1$.

The characteristic equation of the iterated corrector (2.1) is

$$P_h(z) = \sum (a_i + h^2 g\, b_i) z^{k-i} = 0 \tag{2.3}$$

where $g = \partial f/\partial y$, is considered constant. The polynomial $P_0(z)$ has a double root of 1. Call the corresponding roots for $h \neq 0$, r_0 and r_1. In analogy with the first-order case, the principal roots approximate the basic exponential solutions of the differential equation $y'' = gy$, to some order of accuracy. In a manner similar to that of Henrici [1], it can be shown that if (2.1) is of order p,

$$r_i = e^{\pm\sqrt{g}\,h} + O(h^{p+1})$$

$$r_i^n = e^{\pm\sqrt{g}\,(t_n - t_0)} + O(h^{p+1}), \qquad i = 0,1 \;.$$

The Propagated Error Estimate

The solution to the difference equation

$$\sum (a_i + \bar{h}\, b_i) e_{n-i} = 0$$

for the error $e_n = y_n - y_n^*$ (where $\bar{h} = h^2 g$ and where distinct roots r_i of the characteristic polynomial are assumed) is

$$e_n = \sum^{k-1} c_i r_i^n - \frac{T_n}{\bar{h} \Sigma b_i} . \tag{2.4}$$

The true solution $y_n = y(t_n)$ is, as usual, assumed to satisfy

$$\sum a_i y_{n-i} + h^2 \sum b_i \ddot{y}^*_{n-i} = T_n$$

and the computed solution

$$\sum a_i y^*_{n-i} + h^2 \sum b_i \ddot{y}_{n-i} = R_n$$

where T_n and R_n are local truncation and round-off errors respectively. In (2.4) the latter is neglected as it is in the remainder of the paper.

The error estimate is derived by solving for the coefficients c_i in (2.3). The initial errors $e_0, e_1, \ldots, e_{k-1}$ are assumed to be zero and the truncation errors constant. The (non-singular system)

$$\sum^{k-1} c_i r_i^n = \frac{T}{\bar{h} \Sigma b_i} , \qquad n = 0,1,\ldots,k-1$$

results and the principle coefficients have the determinantal solutions

$$c_0 = \frac{T}{\bar{h}\,\Sigma\, b_i}\,\frac{D_0}{D} \quad \text{and} \quad c_1 = \frac{T}{\bar{h}\,\Sigma\, b_i}\,\frac{D_1}{D} .$$

Assume the stability condition: The extraneous roots $r_2, r_3, \ldots, r_{k-1}$ of $P_0(z)$ satisfy $|r_i| \leq 1$ and are of multiplicity ≤ 2 if on the unit circle. The error (2.4) can then be approximated by

$$e_n \sim \frac{T}{\bar{h}\,\Sigma\, b_i} \left(\frac{D_0}{D}\, r_0^n + \frac{D_1}{D}\, r_1^n - 1 \right) .$$

Replacing the principle roots appearing explicitly in the approximation by the appropriate exponentials, we have, for small h,

$$e_n \sim \frac{T}{\bar{h}\,\Sigma\, b_i} \left[\frac{D_0 + D_1}{D}\, \frac{e^{h\sqrt{g}(t_n - t_0)} + e^{-h\sqrt{g}(t_n - t_0)}}{2} \right.$$

$$\left. + \frac{D_0 - D_1}{D}\, \frac{e^{h\sqrt{g}(t_n - t_0)} - e^{-h\sqrt{g}(t_n - t_0)}}{2} - 1 \right] .$$

Let

$$\bar{D}(z_1,z_2) = \begin{vmatrix} 1 & 1 & 1 & \cdots\cdots & 1 \\ z_1 & z_2 & r_2 & \cdots\cdots & r_{k-1} \\ z_1^2 & z_2^2 & r_2^2 & \cdots\cdots & r_{k-1}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ z_1^{k-1} & z_2^{k-1} & r_2^{k-1} & \cdots\cdots & r_{k-1}^{k-1} \end{vmatrix} .$$

Then $D_0 = \bar{D}(1,r_1)$; $D_1 = (r_0,1)$ and $D = \bar{D}(r_0,r_1)$. Let $d_0 = \bar{D}(1,e^{-\sqrt{gh}})$, $d_1 = \bar{D}(e^{\sqrt{gh}},1)$ and $d = \bar{D}(e^{\sqrt{gh}},e^{-\sqrt{gh}})$.

The remaining analysis and applications concern the case in which extraneous roots of $P_0(z)$ are distinct and non-zero. Define $q(z) = \bar{D}(z,1)$. For sufficiently small $h, q(z)$ has a simple root at $z = 1$. Therefore $q'(1) \neq 0$ Since $\bar{D}(e^{\sqrt{gh}},e^{-\sqrt{gh}}) = \bar{D}(e^{\sqrt{gh}} - e^{-\sqrt{gh}},e^{\sqrt{gh}})$

$$\lim_{h\to 0} \frac{d}{h} = \ell, \qquad 0 < |\ell| < \infty .$$

Similarly

$$\frac{d_1 + d_0}{2h} \longrightarrow \ell .$$

Define α_0 to be equal to (the finite limit)

$$\lim_{h\to 0} \frac{d_1 + d_0}{2h\bar{d}} .$$

We then obtain the error estimate

$$e_n = \frac{T}{\bar{h}\,\Sigma\, b_i}\left[\frac{e^{\sqrt{\bar{h}}(t_n-t_0)}+e^{-\sqrt{\bar{h}}(t_n-t_0)}}{2} + \alpha_0\,\bar{h}\,\frac{e^{\sqrt{\bar{h}}(t_n-t_0)}-e^{-\sqrt{\bar{h}}(t_n-t_0)}}{2} - 1\right] . \tag{2.5}$$

The formula is valid with the assumptions on the extraneous roots noted above. In the construction of efficient Class II correctors, we are primarily interested in reducing the magnitude of the factor $T/(\bar{h}\,\Sigma\, b_i)$ which appears in the error estimate. A similar estimate involving the predictor is derived later, in the consideration of optimum algorithms.

3. REDUCTION OF PROPAGATED ERROR BY PLACEMENT OF EXTRANEOUS ROOTS OF $P_0(z)$

Application of the second and third order equations (2.2) to the difference equation (2.1), yields the equation

$$\begin{aligned} 2 \sum b_i &= - \sum i^2 a_{k-i} \\ &= - \sum (k - i)(k - i - 1)a_i \ . \end{aligned}$$

Here, h has been taken to be equal to 1 and $t_{n-k} = 0$. $2 \Sigma b_i$ is seen to be equal to $- P_0''(z)$ evaluated $z = 1$. If

$$- P_0(z) = (z - 1)^2 (z - r_2) \cdots (z - r_{k-1})$$

then

$$- P_0''(1) = 2(1 - r_2)(1 - r_3) \cdots (1 - r_{k-1}) .$$

The result for Class II operators follows, which is analogous to that given by Hull and Newbery [2]:

THEOREM 3.1. The sum of the b_i in the difference equation (2.1) is equal to the product of the distances of the extraneous roots of $P_0(z)$ from the point 1 in the complex plane.

In connection with the construction of efficient Class II correctors, a result concerning the order of the difference equation (2.1) is of interest and of some importance. The following theorem is implied by theorems of Henrici [1, Ch. 6] involving the Taylor expansion of

$$\frac{P_0(z)}{(\log z)^2}$$

about the point 1 and the mapping of the interior of the unit circle

onto the left half-plane. It is convenient to prove the theorem directly as an extension of the Class I result [3].

THEOREM 3.2. <u>Let the coefficients</u> a_i, b_i <u>in</u> (2.1) <u>satisfy the first</u> $k+3$ <u>of the order equations</u> (2.2); <u>if</u> k <u>is even and if the extraneous roots of</u> $P_0(z)$ <u>satisfy</u> $|r_i| = 1$, $r_i \neq 1$, $i = 2,3,\ldots,k-1$, <u>then</u> (2.1) <u>is of order</u> $k+2$.

<u>Proof</u>. The term $C_{k+3}\, y^{(k+3)}\, h^{k+3}$ in the expansion of (2.1) as operator will be shown to be zero. C_{k+3} satisfies

$$-(k+1)!\, C_{k+3} + \sum^{k-1} i^{k+3} a_{k-i} + (k+3)(k+2) \sum^{k-1} i^{k+3} b_{k-i} = 0$$

or for $R = -(k+1)!\, C_{k+3}$

$$\sum^{k-1} i^{k+3} a_{k-i} + (k+3)(k+2)\left(\sum^{k-1} i^{k+1} b_{k-i} + R \right) = 0 \quad . \qquad (3.3)$$

Equation (3.3) together with the order equations satisfied, with the exception of the first three, give $k+1$ equations in the $k+1$ unknowns $R, b_0, b_1, \ldots, b_{k-1}$. R can be written as the quotient of two determinants. The denominator is

$$D = \begin{vmatrix} k & k-1 & \cdots\cdots & 1 \\ k^2 & (k-1)^2 & \cdots\cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ k^k & (k-1)^k & \cdots\cdots & 1 \end{vmatrix}$$

and the numerator is

$$\begin{vmatrix} S_1 & k & k-1 & \cdots\cdots & 1 \\ S_2 & k^2 & (k-1)^2 & \cdots\cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{k+1} & k^{k+1} & (k-1)^{k+1} & \cdots\cdots & 1 \end{vmatrix}$$

where

$$-S_j = \frac{1}{(j+1)(j+2)} \sum^{k-1} i^{j+2} a_{k-i} \;.$$

The polynomial $\pi(x) = x(x-1) \cdots (x-k)$ can clearly be written in the form

$$\frac{1}{D} \begin{vmatrix} x & k & k-1 & \cdots\cdots & 1 \\ x^2 & k^2 & (k-1)^2 & \cdots\cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x^{k+1} & k^{k+1} & (k-1)^{k+1} & \cdots\cdots & 1 \end{vmatrix} \;.$$

If $\pi(x) = x^{k+1} + c_1 x^k + \cdots + c_{k+1} x$, it follows that

$$-R = S_{k+1} + c_1 S_k + \cdots + c_{k+1} S_1 .$$

It is clear that

$$\int_0^i \int_0^y \pi(x)dx\,dy$$

$$= \frac{i^{k+3}}{(k+2)(k+3)} + \frac{c_1 i^{k+2}}{(k+1)(k+2)} + \cdots + \frac{c_{k+1} i^3}{6}$$

and

$$R = \sum^{k-1} a_i \int_0^{k-1} \int_0^y \pi(x)dx\,dy$$

$$= \sum^{k-1} a_i + a_{i+1} + \cdots + a_{i-1} \int_{k-i-1}^{k-1} \int_0^y \pi(x)dx\,dy$$

$$= \sum^{k-1} \bar{a}_i J_i .$$

Now

$$\frac{\Sigma a_i z^{k-i}}{(z-1)} = \bar{a}_{k-1} + \bar{a}_{k-2} + \bar{a}_{k-i} + \cdots + \bar{a}_0 z^{k-1}$$

so that the $\bar{a}_i$ are coefficients of the polynomial whose roots are the extraneous roots, plus one principal root, or the characteristic polynomial $P_0(z)$.

If k is even $J_i = J_{k-i-1}$, since $\pi(k - x) = (-1)^{k+1} \pi(x)$ and

$$F(y) \equiv \int_0^y \pi(x)dx$$

satisfies $F(y) = F(k - y)$.

R can be made to vanish if $\bar{a}_i = - \bar{a}_{k-i-1}$. This can be accomplished by taking the extraneous roots on the unit circle; for then, the coefficients of the polynomial

$$P = \sum_{i=0}^{k-2} c_i z^{k-i-2}$$

having the extraneous roots of $P_0(z)$ as its roots, satisfy $c_i = c_{k-i-2}$. Multiplication by $(z - 1)$ gives the desired result.

In $k + 1^{th}$ order methods to be constructed, extraneous roots of $P_0(z)$ are less than 1 in magnitude, with fixed arguments on circles of variable radius r. It can be expected that the primary error coefficient, c_{k+3}, of the methods will approach zero as r

approaches 1.

One can hope to move the extraneous roots of $P_0(z)$ from the origin westward, sufficiently far for the increase in Σb_i to be effective before stability is too severely impaired. Likewise, a decrease in local truncation error is possible, at least for even k, if the roots can be pushed close enough to the unit circle. One can, with this procedure, reduce $|T/(\Sigma b_i)|$ and the propagated error e_n given by the estimate (2.5). Intuitively, a loss of stability is expected. The error estimate, as given, does not apply to the Cowell method. It does, however, apply to methods with distinct non-principal roots arbitrarily close to the origin.

It should here be recalled that the estimate (2.5) is based on a stability analysis applied to a single linear equation. Moreover, it applies to the iterated corrector. If the corrector is not iterated to convergence, the error estimate will, of course, involve the coefficients of the predictor equation as well as the truncation error of the predictor. Estimates of this type are derived in Section 5 where it is seen that these quantities will be multiplied by a power of h according to the number of corrections, and can therefore be neglected in the consideration of truncation error reduction. The computations in the next section show that methods based on this analysis exist which are significantly superior to the Cowell corrector, on an orbit problem, at all step-sizes, even for a single correction (algorithm PEC-Störmer

predictor). It will be shown that the stability interval or optimized methods used with the PEC algorithm can be enlarged beyond that associated with the Cowell corrector. In general, the error advantage of optimized methods (within stability regions) over the Cowell method is more marked in the $P(EC)^2$ than in the PEC algorithm. The broader consideration of efficiency and the number of corrections is discussed in Section 5.

Finally, it is noted that Σb_i for constructed correctors is bounded by 2^{k-2} (for k^{th} difference method). Extraneous roots associated with the Cowell corrector are, of couse, zero, which implies $\Sigma b_i = 1$.

4. COMPUTATIONAL ANALYSIS

The computational results are concerned with details of construction of new methods, a linear stability analysis of new methods, and computations of a circular orbit for error analysis.

Construction of Corrector Methods

The construction and analysis is restricted to 10^{th} difference corrector methods. In all cases, these methods are used with the same predictor -- the 11^{th} order Störmer formula.

Methods are derived (as in [3]) by specifying extraneous roots of $P_0(z)$. This determines the a_i in (2.1). The b_i are determined by requiring order $k+1 = 11$. Two classes of methods are considered. These are here called

(1) Westward (W) methods, and

(2) Restricted Westward (RW) methods.

The (eight) extraneous roots of each method have the same moduli. Each method is determined by specifying the modulus r and a set of eight θ values. The θ values for W methods are

$$\pm\theta = \{90°, 115°, 140°, 165°\} .$$

For the RW methods,

$$\pm\theta = \{140°, 150°, 160°, 170°\} .$$

The set of moduli considered in both classes of methods is r: .1 (.1) .9. Note that $R(0) = RW(0) =$ Cowell method.

Stability

For each corrector (C), the characteristic polynomials corresponding to both algorithms PEC and $P(EC)^2$ were derived with the Störmer predictor (P). The stability, as applied to the solution of the equation $\ddot{y} = \lambda y$ was examined by computing the roots of the char-

acteristic equation at step-size shown in the graphs.

Figures 1 and 2 show comparative stability behavior for selected methods and algorithm $P(EC)^2$. For reference, the abscissa gives the step-sizes relative to the values $\lambda = 1$ (period 2π) and the value $\lambda = .77825 \times 10^{-6}$ (a circular orbit of radius 8×10^{6} meters, period, approximately 2 hours). The magnitude of the largest extraneous root is plotted against the step-size.

Figures 1 and 2 show expected behavior: the stability intervals of both RW and W methods are considerably reduced from that of the Cowell method. It should be pointed out, however, that the Cowell method is over-stable for many applications in a sense to be made clear later. RW(.3) and W(.7) are adequately stable for most applications and, as will be seen, considerably more accurate than the Cowell method. It is noted, finally, that the W formulas permit placement of the roots much closer to the unit circle than do the RW formulas, for the same stability interval. This will be significant in the reduction of propagated error.

Figures 3 and 4 display the comparative stability (linear analysis) of selected methods with algorithms PEC. Of interest here, is the fact that the stability interval of optimized methods can actually be increased over that of the Störmer-Cowell algorithm, with proper selection of the modulus r. It will also be seen that propagated

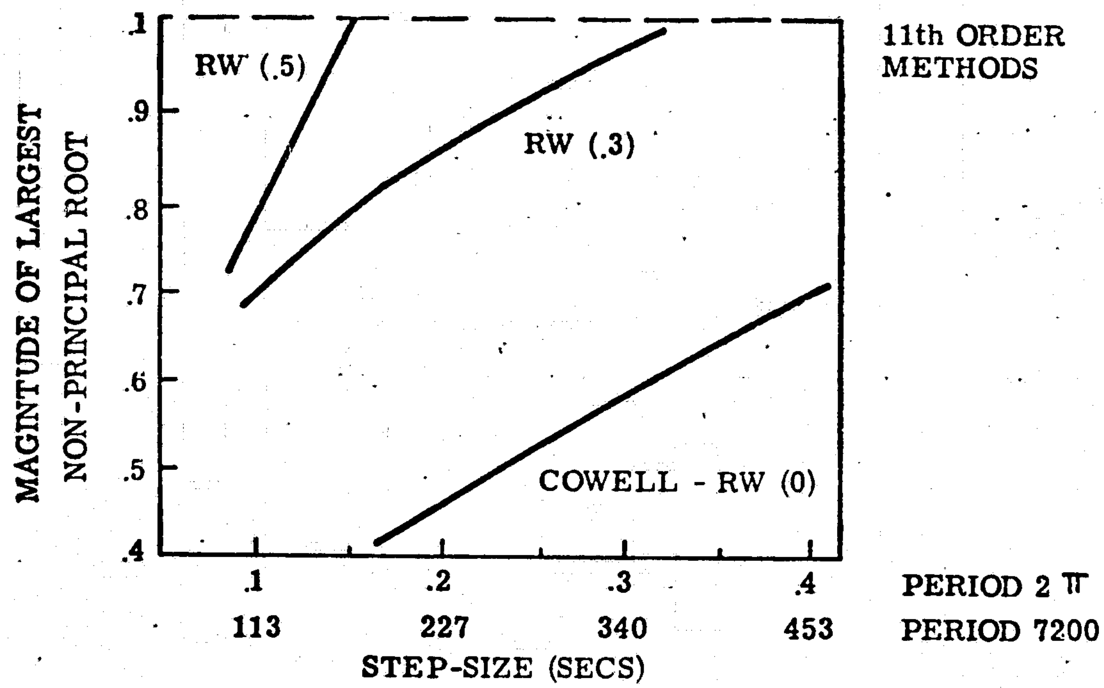


Fig. 1 LINEAR STABILITY COMPARISON P(EC)^2

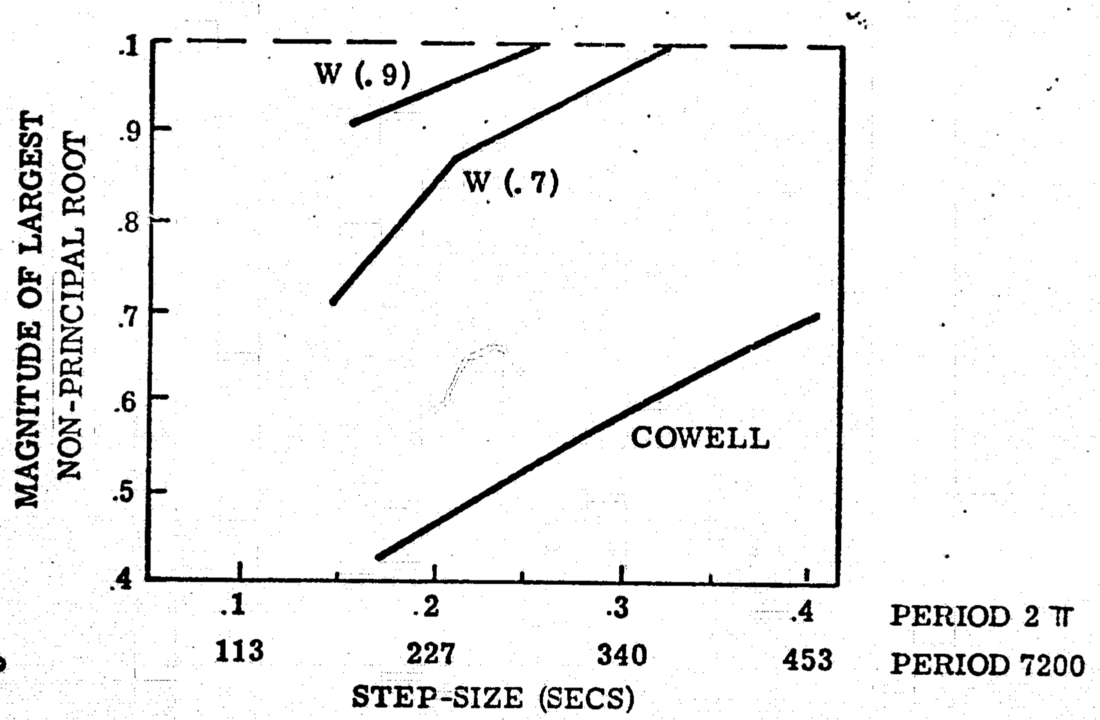


Fig. 2 LINEAR STABILITY COMPARISON P(EC)^2

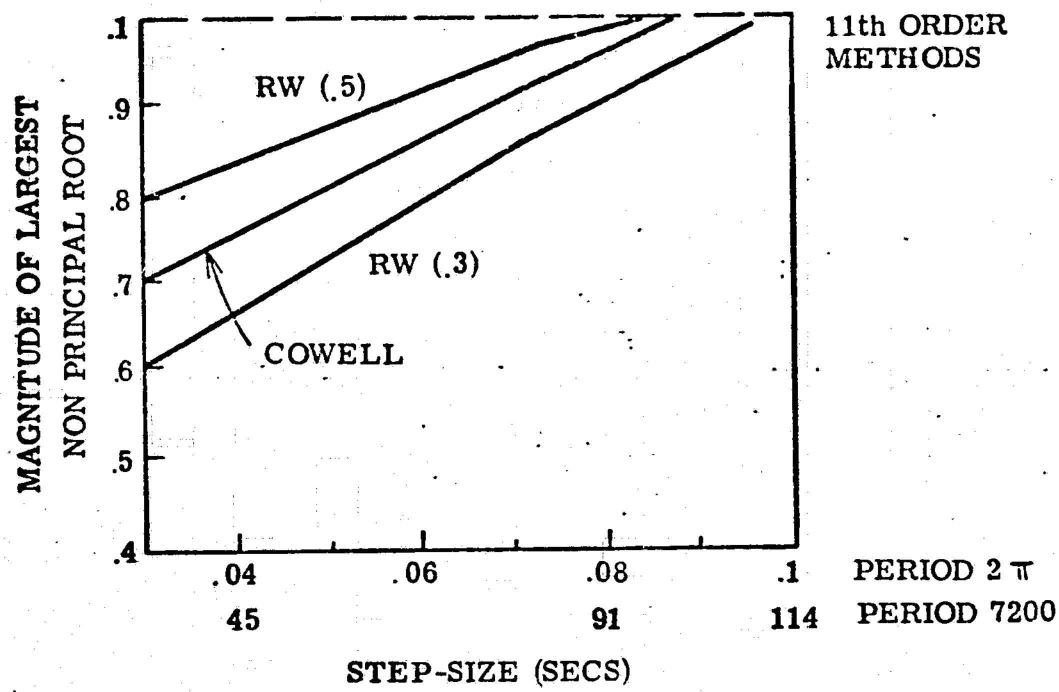


Fig. 3 LINEAR STABILITY COMPARISON PEC

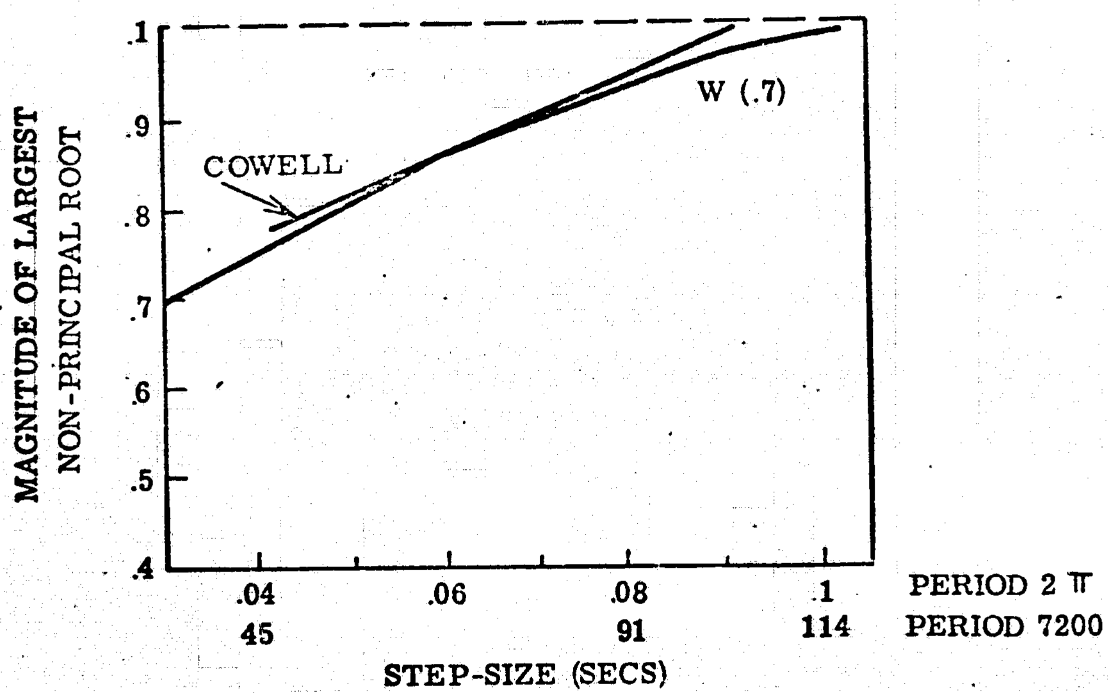


Fig. 4 LINEAR STABILITY COMPARISON PEC

error in both RW(.3) and W(.7) is reduced in comparison with the Störmer-Cowell error. RW(.2), RW(.4), W(.6) and W(.8) (not shown) also have larger stability intervals than the Cowell method.

Propagated Error

The methods considered above were used to solve the initial value problem

$$\ddot{\bar{x}} = - \frac{\mu \bar{x}}{|\bar{x}|^3}, \qquad \mu = \text{constant}$$

with initial values corresponding to a circular orbit with a 7200 sec. period.

Figure 5 shows the error (length of error vector where the analytical solution is used as reference) after 84 revoluations of integration, using the algorithm $P(EC)^2$. Results of the best observed W method and RW method are shown. The Störmer-Cowell algorithm, although stable at step-sizes much greater than 300 secs., produces rather severe error beyond that step-size. Algorithms RW(.3) and W(.7) are less stable, but considerably more accurate in the step-size domain shown.

Figures 6 and 7 show the error in the solution generated by algorithm P(EC) with selected new methods. In these two figures in-

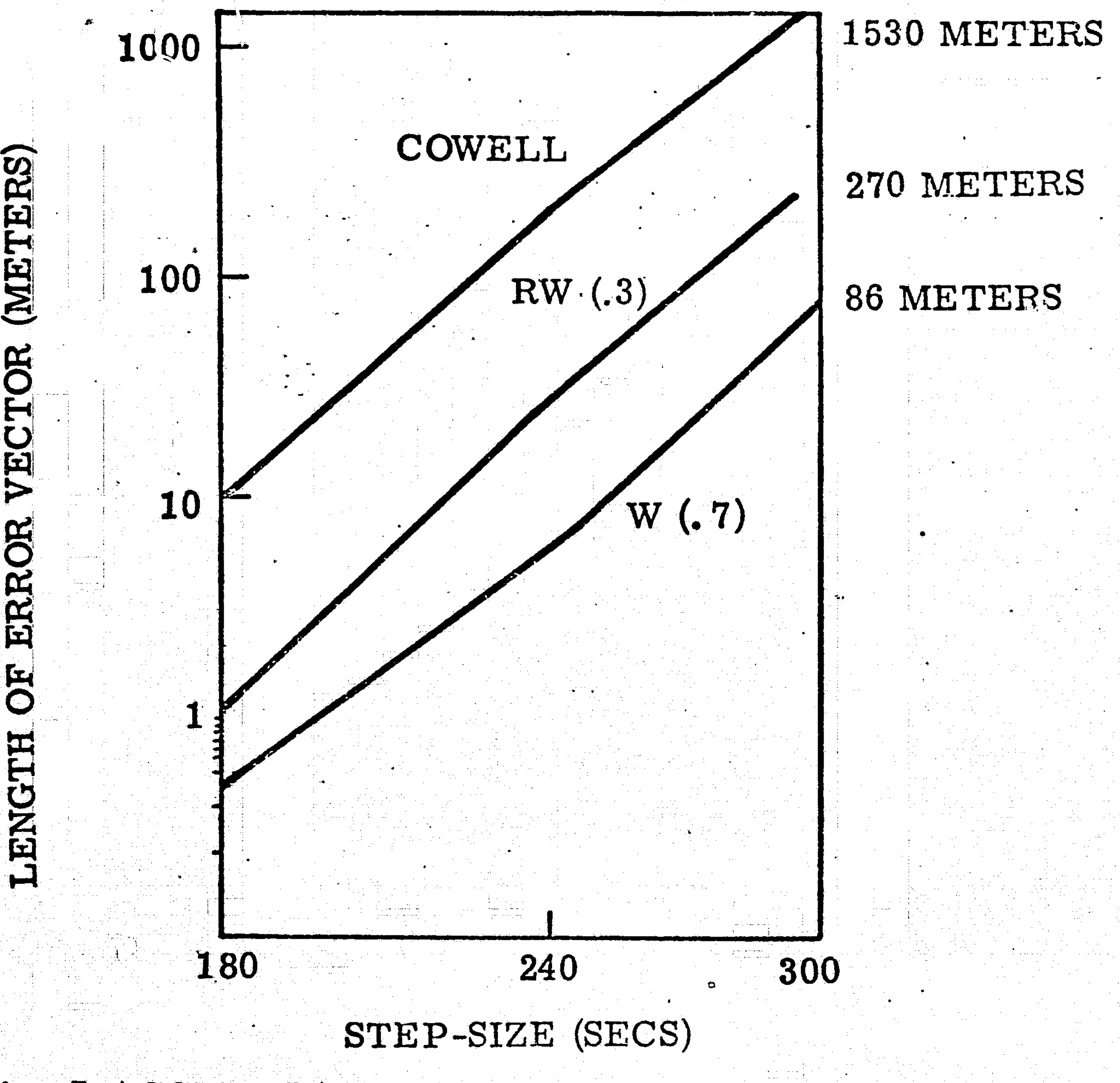


Fig. 5 ACCUMULATED ERROR AFTER 84 REVOLUTIONS (2-HR ORBIT) $P(EC)^2$

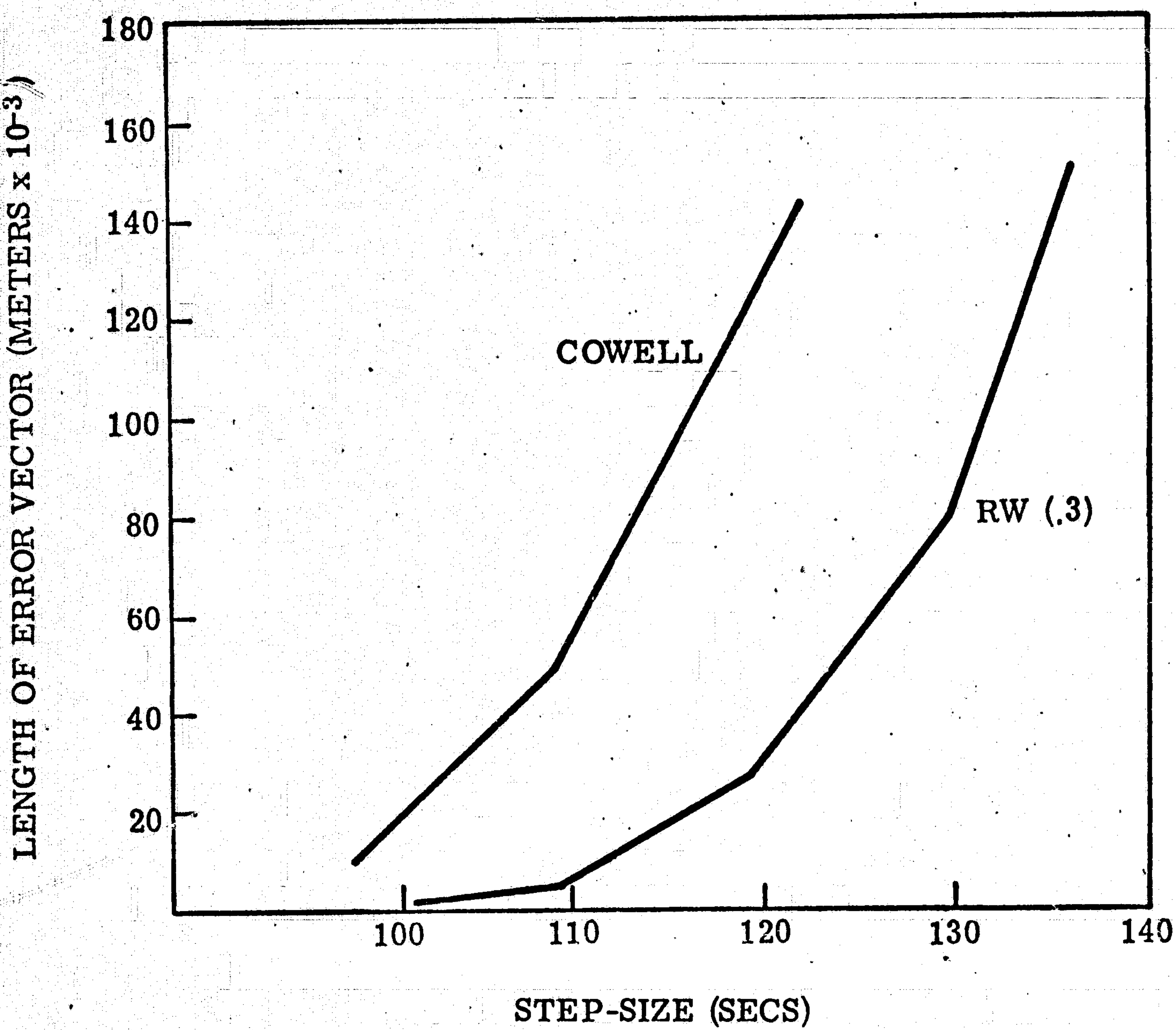


Fig. 6 ACCUMULATED ERROR AFTER 84 REVOLUTIONS (2-HR ORBIT) PEC

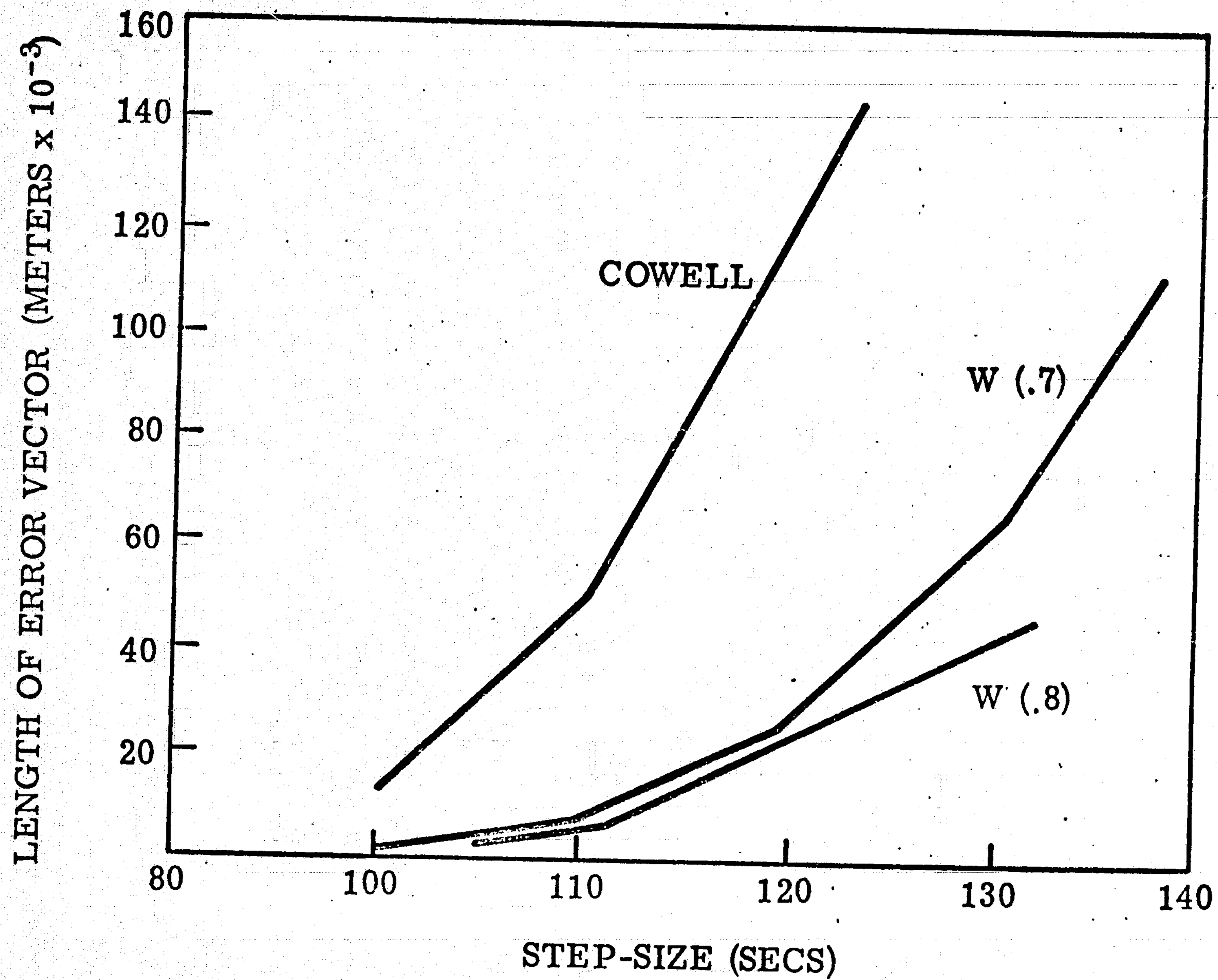


Fig. 7 ACCUMULATED ERROR AFTER 84 REVOLUTIONS (2-HR ORBIT) PEC

stability occurs approximately where the graphs terminate. Thus, we can see that there exists a variety of easily obtainable methods which are both more stable and more accurate than the Cowell method when used with these algorithms on the orbit problem. W(.7) is the most stable of the methods considered. RW(r) methods, $r \geq .5$ perform less satisfactorily than the Cowell method at stepsizes greater than 100 secs. and are unstable at $h > 110$ secs. in the problem considered.

The actual stability interval is somewhat larger for most methods than that indicated by the linear stability analysis, as can be seen by comparing with Figures 3 and 4. However, valid comparisons (for this problem) can be made on the basis of the linear analysis.

The computations were done in double precision Fortran on the IBM 360/50.

5. OPTIMUM EFFICIENCY ALGORITHMS

Before concluding, it is remarked that in a fashion similar to that given by Hull and Creemer [5], one can extend the error estimate (2.5) to include the effects of the predictor for the purpose of investigating optimum efficiency $P(EC)^m$ or $PE(CE)^m$ algorithms for Class II methods. In particular, if we denote the predictor by

$$\sum a_i^* y_{n-i} + h^2 \sum b_i^* y''_{n-i} = 0, \quad b_0^* = 0; \; a_0^* = -1$$

with truncation error T^*, then we can derive the following estimator:

$$e_n \sim \alpha(T,T^*,m)[c_0 r_0^n + c_1 r_1^n - 1]$$

where for $P(EC)^m$,

$$\alpha(T,T^*,m) = \frac{T}{\bar{h}\,\Sigma\,\beta_i}\left(1 - \sigma_1(m)\,\bar{h}\,\Sigma\, b_i^*\right) + T^*\,\sigma_1(m)$$

$$\sigma_1(m) = \frac{(\bar{h}\,b_0)^{m-1}\,(\bar{h}\,b_0 - 1)}{(\bar{h}\,b_0)^m - 1}$$

and for $PE(CE)^m$

$$\alpha(T,T^*,m) = \frac{T + \sigma_2(m)T^*}{\bar{h}\,\Sigma\,\beta_i + \sigma_2(m)\,\Sigma\, b_i^*}$$

$$\sigma_2(m) = \frac{(\bar{h}\,b_0)^m\,(\bar{h}\,b_0 - 1)}{(\bar{h}\,b_0)^m - 1} \; .$$

From these estimates, it can be argued that from the point of view of

minimizing the number of function evaluations required to achieve a given accuracy, $m = 1$ is generally best in both $P(EC)^m$ and $PE(CE)^m$ type algorithms. It is noted, however, that the PEC method is highly unstable -- particularly in the Cowell method, as seen in the numerical results -- so that the most useful methods from the point of view of stability are $P(EC)^2$ or PECE.

6. SUMMARY AND REMARKS

A summary of approximate errors in the solution generated by methods RW(.3) and W(.7) is given in comparison to that of the Cowell method.

APPROXIMATE ERROR RATIOS

	Algorithm PEC				Algorithm $P(EC)^2$		
h (secs.)	100	110	120	130	180	240	300
$\frac{e_n(0)}{e_n[RW(.3)]}$	3.8	5.8	4.2	∞	8.0	6.8	5.5
$\frac{e_n(0)}{e_n[W(.7)]}$	5.0	6.1	5.1	∞	30.0	36.1	17.9

The algorithm $P(EC)^2$ reflects more accurately the effect of

the optimization, since the attempt has been only to optimize the correctors. The approximate values Σb_i and average error reduction for the three methods above are:

Method	Σb_i	Average of above error ratios, $P(EC)^2$
Cowell	1.0	-
RW(.3)	6.9	6.8
W(.7)	22.6	28.0

Some effect, due to Theorem 3.2, is probably observable at the smallest above step-size, and perhaps, more generally in W(.7). Computation shows that it is necessary to take the modulus r close to 1 to achieve an appreciable reduction of the primary truncation error coefficient (C_{13} in the expansion $\Sigma\, C_i\, y^{(i)}\, h^{(i)}$ of (2.1) in operator form). Stability of the RW methods is severely degraded before this occurs. The theorem concerns only the single mentioned error coefficient. Small step-sizes de-emphasize the predictor error, and emphasize the reduced primary error coefficient.

The described methods are closely related to the Cowell method. They are almost as easily implemented, as is the latter method, and the running time is about the same. Moreover, they can readily be

written in summed form as variant Gauss-Jackson methods. Again, this implementation involves little more work.

Stability of many new methods used with algorithm $P(EC)^2$ is undoubtedly adequate for most applications. Unutilizable stability of the Cowell method has been traded for increased accuracy. As pointed out, stability of algorithm PEC can be improved, as well as accuracy. $P(EC)^n$ can be expected to show more improved performance for $n > 2$.

Algorithm PECE produces similar results. For example, relative to the same orbit problem used throughout the paper, the accumulated errors (meters) are given in the following table.

Step-Size (secs.)	180	240	300
Cowell Method	8.3	163.0	1398.7
Method W(.7)	.3	6.4	64.5

The generalized mean value theorem can be applied [1] to the Cowell method, yielding an estimate of local truncation error in terms of the primary error coefficient mentioned above. Without further analysis, this theorem cannot be used to estimate, in the same manner, the local error in the new methods. The application

of this theorem is to be investigated.

Finally, only a rudimentary optimization procedure has been used, thus far in this work. More sophisticated techniques can no doubt yield better methods, although there is a well defined limit to

$$\min \frac{T}{|\Sigma\, b_i|}\ ,$$

which has been found by crude experimentation to be roughly of order 10.

7. REFERENCES

[1] Henrici, P. Discrete Variable Methods in Ordinary Differential Equations. Wiley, New York, 1961.

[2] Hull, T.E. and A.C.R. Newbery, "Integration Procedures Which Minimize Propagated Errors," SIAM JOURN. NUM. ANAL. 9 (1961): 31-47.

[3] ________________________, "Corrector Formulas for Multi-Step Integration Methods," SIAM JOURN. NUM. ANAL. 10 (1962): 351-369.

[4] Hull, T.F. and Creemer, A.L., "Efficiency of Predictor-Corrector Procedures," J. ACM, 40 (1963): 291-301.

[5] Ralston, A. A First Course in Numerical Analysis. McGraw-Hill, New York, 1965.

APPENDIX B

ON THE NON-EQUIVALENCE OF MAXIMUM POLYNOMIAL DEGREE NORDSIECK-GEAR AND CLASSICAL METHODS

by

Roy Danchick

INTRODUCTION

The intent of this paper is to show that maximum polynomial degree (p.d.) Nordsieck-Gear methods (References 2 and 3) are not equivalent, in the sense of Descloux and Gear (References 1 and 2), to corresponding Adams-Moulton methods. The notion of derivative interpolating polynomials is introduced and used to derive both the correct initial step polynomial degree maximizing coefficients and explicit, initial step truncation-error formulas. The associated error constants are shown to be vanishingly smaller than their classical analogues in the sense that the ratio of the maximal polynomial degree Nordsieck-Gear initial step error constant to the equal polynomial degree Adams-Moulton error constant tends to zero as the polynomial degree tends to infinity. Most importantly, it is shown that even with correct initial step polynomial degree maximizing coefficient, non-trivial stabilizing correction of retained information induces a reduction of one in method polynomial degree after the initial step, if the degree maximizing coefficient is held constant. Then, a generalized "error mop-up" theorem follows. The theorem's conclusion provides the foundation for a new method based on the recursive computation of the p.d. maximizing coefficients. The use of these varying coefficients preserves maximal degree in the solution throughout the numerical integration from starting through all intermediate stepsize changes. This preservation of maximal p.d. by coefficient variation distinguishes the new method from the standard Nordsieck-Gear methods. Lastly, though exact scaled derivative starting values are postulated in the proofs, all conclusions remain valid if, instead, starting errors of order $O(h^{k+2})$ in each scaled derivative are assumed. A starting procedure which guarantees this order of accuracy has been developed and will be described in a forthcoming paper.

DISCUSSION

Preliminaries

The k-predictor/(k+1)-corrector version of the Nordsieck (k+1)-value method for the numerical integration of the p^{th} order differential equation,

$$y^{(p)}(x) = f\,(x, y, y^{(1)}, \ldots, y^{(p-1)})$$

can be written in Gear's compact notational form as the vector difference equation:

$$(1) \qquad Y_{n+1} = AY_n + \ell\left\{(h^p/p!)f((n+1)h,\ AY_n) - e_p^{\,T}AY_n\right\}$$

where
h = step size
$Y_n = \left\{y(nh),\ hy^{(1)}(nh), \ldots,\ h^k y^{(k)}(nh)/k!\right\}^T$,
A = the (k+1) x (k+1) Pascal Triangle Matrix,
e_p = the vector whose p+1 component is one with all other components zero.

The vector $\ell = (\ell_0, \ell_1, \ldots, \ell_k)^T$ is chosen so that

(a) The matrix $(I - \ell e_p^{\,T})A$ has characteristic equation $p(x) = (x-1)^p x^{k+1-p}$

and (b) The relation defined by Equation (1) is exact in the first p+1 components for all polynomials of degree $\leq$ k+1, given the exact values for the components of Y_0 and the exact value of $y^{(p)}(h)$.

In this formulation, abbreviated here as (k, p), Gear has shown that p(x) is independent of $\ell_0, \ell_1, \ldots\ldots \ell_{p-1}$; thus, these parameters can be chosen to maximize polynomial degree in the sense of (b) above.

The method is optimally stable in the sense of (a) above, and the initial step truncation error is $O(h^{k+2})$ in each of the first p+1 components of $Y_0(h)$ if $y \in C^{k+2}[0,h]$ and exact starting values are given. Gear's insight into the stabilizing effect of properly correcting all retained information, and his constructive proof of the existence of a stabilizing ℓ for each k, has established a basis for hurdling the Dahlquist stability problem. Unfortunately, however, a logical oversight by the late Doctor Nordsieck in his trailblazing paper, has led to some logically incorrect conclusions about the Nordsieck-Gear Methods. The main purpose here is to lay to rest these mistaken, yet widely held, notions and

to reveal a unique family of truly one-step, maximum-order, optimally stable methods.

A Counterexample

Gear's choice of ℓ_0 (Y in Nordsieck's notation) does not actually maximize the initial step polynomial degree of the method if the k+1 dimensional vector of retained information is $(y(x), hy^{(1)}(x), h^2y^{(2)}(x)/2!\ldots, h^ky^{(k)}(x)/k!)$. This can be shown by the simple counterexample, $y(x) = x^3$, in the evaluation of Nordsieck's corrector form for k = 2:

$$y(x+h) = y(x) + h\left\{f(x) + a(x) + \frac{5}{12}\left[f(x+h) - f_p(x+h)\right]\right\} \tag{2}$$

where
$$f(x) = y^{(1)}(x)$$
$$a(x) = hy^{(2)}(x)/2$$
$$f_p(x) = f(x) + 2a(x).$$

For $y(x) = x^3$, when $y(x)$, $y^{(1)}(x)$, $y^{(2)}(x)$, and $y^{(1)}(x+h)$ are given exactly, the right-hand side (R.H.S.) of Equation 2 above can be written:

$$\begin{aligned} & x^3 + h\left\{3x^2 + 3hx + \frac{5}{12}[3(x^2 + 2hx + h^2) - (3x^2 + 6hx)]\right\} \\ &= x^3 + h\left\{3x^2 + 3hx + \frac{5}{4}h^2\right\} \\ &= (x+h)^3 + h^3/4 \neq (x+h)^3 \end{aligned} \tag{3}$$

The logical inconsistency in Equation (3) occurs because the coefficient Y = 5/12 is inappropriate for the initial step maximization of polynomial degree. The transformation of predictor-basis matrices, such as the Pascal triangle, though preserving the roots of characteristic stability polynomial at h = 0, does not, in general, preserve maximal initial step polynomial degree. The correct initial step polynomial degree maximizing coefficients for Gear's p = 1, and p = 2 (k+1)-value (k scaled derivatives) methods are, for example:

(4)

p = 1	p = 2
$\ell_0 = 1/(k+1)$	$\ell_0 = 2/[k(k+1)]$
	$\ell_1 = 2/k$

To vividly contrast the effects of using the correct and the incorrect initial step coefficient, consider the initial value problem

$$(5)\qquad \begin{aligned} y^{(1)}(x) &= 3y/x \\ y(x_0) &= x_0^{\,3} \\ &\left(|h|<|x_0|,\ x_0 \neq 0\right) \end{aligned}$$

whose solution is simply $y(x) = x^3$. If, starting with the exact value of $6x_0$ for $y_0^{(2)}(x\ )$, the Nordsieck-Gear 3-value method is iterated, then

$$(6)\qquad \begin{aligned} y_0(x_0+h) &= x_0^{\,3}+h\left\{3x_0^{\,2}+3hx_0\right\} = (x_0+h)^3-h^3 \\ y_1(x_0+h) &= (x_0+h)^3 + h^3/4 \end{aligned}$$

$$(7)\qquad \begin{aligned} y_2(x_0+h) &= x_0^{\,3} \quad +h\left\{3x_0^{\,2}+3hx_0+\frac{5}{12}\left[\frac{3\ \left[(x_0+h)^3+h^3/4\right]}{(x_0+h)} - 3x_0^{\,2}-6hx_0\right]\right\} \\ &= x_0^{\,3}+h\left\{3x_0^{\,2}+3hx_0+\frac{5}{4}\left[h^2+h^3/4(x_0+h))\right]\right\} \\ &= (x_0+h)^3 + h^3/4 + 5h^4/\left[16(x_0+h)\right] \end{aligned}$$

It readily follows, by induction on n, that

$$(8)\qquad y_n(x_0+h) = y(x_0+h) + (h^3/4)\sum_{i=0}^{n} (\bar{h})^i$$

where

$$\bar{h} = 5h/\left[4(x_0+h)\right]$$

By contrast, using the correct coefficient (1/3),

$$(9)\qquad \begin{aligned} y_0^*(x_0+h) &= (x_0+h)^3 - h^3 \\ y_1^*(x_0+h) &= x_0^{\,3}+h\left[3x_0^{\,2}+3hx_0+\frac{1}{3}\left\{\frac{3\ \left[(x_0+h)^3-h^3\right]}{(x_0+h)} - 3x_0^{\,2}-6hx_0\right\}\right] \\ &= (x_0+h)^3 - h^4/(x_0+h) \\ y_2^*(x_0+h) &= x_0^{\,3}+h\left[3x_0^{\,2}+3hx_0+\frac{1}{3}\left\{\frac{3\ \left[(x_0+h)^3-h^4/(x_0+h)\right]}{(x_0+h)} - 3x_0^{\,2}-6hx_0\right\}\right] \\ &= (x_0+h)^3 - h^5/(x_0+h)^2 \end{aligned}$$

And, thus, again by induction

(10) $$y_n^*(x_0+h) = (x_0+h)^3 - h^3\left\{h/(x_0+h)\right\}^n$$

Thus, for all $|h|$ sufficiently small

$$y_n^*(x_0+h) \longrightarrow y(x_0+h),$$

but for no $|h|$ does

$$y_n(x_0+h) \longrightarrow y(x_0+h).$$

The Derivative-Interpolating Polynomial

The correct initial-step coefficients can be obtained by inspection from the fundamental, unique, derivative-interpolating polynomials:

(11) $$p_h(t) = T_k(t)+h\left[y^{(1)}(x+h)-T_k^{(1)}(x+h)\right](t-x)^{k+1}\Big/\left[(k+1)h^{k+1}\right]$$

$$(p = 1)$$

(12) $$q_h(t) = T_k(t)+(h^2/2)\left[y^{(2)}(x+h)-T_k^{(2)}(x+h)\right]2(t-x)^{k+1}\Big/\left[(k+1)kh^{k+1}\right]$$

$$(p = 2)$$

where

$T_k(t) = y(x)+y^{(1)}(x)(t-x)+ \ldots . +y^{(k)}(x)\ (t-x)^k/k!$ and

$y^{(i)}(\ )$ is the i^{th} derivative of

$y \in C^{k+2}[x,x+h]$ $(i = 0, 1, 2\ldots, k)$.

By virtue of their construction, p_h and q_h have the following derivative-interpolating properties:

(13)
$$p_h^{(i)}(x) = y^{(i)}(x)$$
$$p_h^{(1)}(x+h) = y^{(1)}(x+h)$$
$$q_h^{(i)}(x) = y^{(i)}(x)$$
$$q_h^{(2)}(x+h) = y^{(2)}(x+h)$$
$$(i = 0, 1, 2, \ldots, k)$$

More generally, the corresponding fundamental interpolating polynomial, $r_h(t)$, for a (k+1)-value direct m^{th} order method, is given by

(14)
$$r_h(t) = T_k(t)+(h^m/m!)\left[y^{(m)}(x+h)-T_k^{(m)}(x+h)\right](t-x)^{k+1}\Big/\left[\binom{k+1}{m}h^{k+1}\right]$$

The derivative-interpolating polynomials $p_h(t)$, $q_h(t)$, and $r_h(t)$ are fundamental in the sense that they not only generate the corresponding polynomial degree maximizing initial-step coefficients but their introduction readily leads to an explicit characterization of the initial-step truncation error in terms of $y^{k+2}(\xi)$ $(x < \xi < x+h)$. As will be shown in the development to follow, the corresponding error constants are $C_1^{k+2} = -1/[(k+1)(k+2)!]$ (p=1) and $C_2^{k+2} = -2/[k(k+2)!]$ (p=2). It should be noted that these error constants are much smaller than those of the corresponding Adams-Moulton and Cowell methods, respectively. For example, for the 5^{th} order Adams-Moulton method the ratio is $\frac{2}{81}$, while for the 5^{th} order Cowell, the ratio is $\frac{1}{6}$.

In fact, it is easy to show that

$$\lim_{k \to \infty} C_1^{k+2}\Big/C_{am}^{k+2} = \lim_{k \to \infty} C_2^{k+2}\Big/C_c^{k+2} = 0$$

where C_{am}^{k+2} and C_c^{k+2} are the equivalent polynomial degree Adams-Moulton and Cowell error constants, respectively.

<u>Derivation of the Initial Single-Step Truncation Error</u>

Suppose $y \in C^{k+2}[x,x+h]$. Let $p_n(t)$ be the derivative-interpolating polynomial introduced in the previous Subsection, "agreeing" with y and its higher order derivatives at t = x and agreeing with $y^{(1)}$ at t = x+h. Let $H_1(t) = [t - x - (k+2)h/(k+1)]\ (t-x)^{k+1}$. $H_1(t)$ is monic, vanishes together with its first k derivatives at t = x, vanishes nowhere else in [x, x+h]. In addition, $H_1^{(1)}(t)$ vanishes at t = x+h.

Let $\bar{t} \in [x,x+h]$. Choose $C(\bar{t}) \ni$: $F(t)=y(t)-p_h(t)-C(\bar{t})H_1(t)$ vanishes at $t = \bar{t}$. Such a choice is always possible as $H_1(t) \neq 0$ in $(x,x+h]$ and $F(x) = 0$ for any choice of $C(x)$. Thus, $F(x) = F(\bar{t}) = 0$.
$\therefore \exists\, \zeta_1 \in (x,\bar{t}) \ni$: $F^{(1)}(\zeta_1) = 0$. Since $F^{(1)}(x+h) = 0, \exists\, \zeta_2 \in (\zeta_1,x+h) \ni$: $F^{(2)}(\zeta_2) = 0$. Since also $F^{(1)}(x) = 0, \exists\, \zeta_2^1 \in (x,\zeta_1) \ni$: $F^{(2)}(\zeta_2^1) = 0$.
Proceeding recursively, we show that $F^{(i)}(t)$ vanishes at least twice in $(x,x+h)$ for $i = 2, 3, \ldots, k-1$. Thus, $F^{(k)}(t)$ vanishes at least three times in $[x,x+h]$. Therefore, $F^{(k+1)}(t)$ vanishes at least twice and hence, finally, $\exists\, \zeta \in (x,x+h) \ni$:

$$F^{(k+2)}(\zeta) = y^{(k+2)}(\zeta) - (k+2)!\; C(\bar{t}) = 0$$

$$\therefore C(\bar{t}) = y^{(k+2)}(\zeta)/(k+2)!$$

In particular, for $\bar{t} = x+h, \exists\, \zeta \in (x,x+h) \ni$:

(15)
$$y(x+h) - p_h(x+h) = y^{(k+2)}(\zeta)H_1(x+h)/(k+2)!$$
$$= [-1/(k+1)]\; y^{(k+2)}(\zeta)\; h^{k+2}/(k+2)!$$

The above proof goes through directly for the $p = 2$ direct second-order method by showing that $G^{(3)}(t)$ vanishes at least three times in $[x,x+h]$, where $G(t) = y(t) - q_h(t) - C(\bar{t})H_2(t)$, $y(t) \in C^{k+2}[x,x+h]$, etc. In this case, $H_2(t) = \dfrac{(t-x-(k+2)h)(t-x)^{k+1}}{k}$ and the $p = 2$ initial single-step truncation error formula is obtained according to

(16)
$$y(x+h)-q_h(x+h) = [-2/k]y^{(k+2)}(\eta)\; h^{k+2}/(k+2)!$$
$$(\eta \in (x,x+h))$$

<u>Reduction and Preservation of Order (Polynomial Degree)</u>

The $(k,1)$ case where $k > 1$ will be covered. The generalization to the case $k > p > 1$ is immediate. Let Y_n, $n=0, 1, 2,\ldots$ be the sequence of approximating vectors for the solution of the differential equation $y^{(1)}(x) = f(x,y)$ whose solution $\bar{y}(x) \in C^{k+2}[0, X]$, and such that $f_y(x,y)$ is continuous in a neighborhood of the graph $(x,\bar{y}(x))$, $x \in [0,X]$. It is further assumed that $Y_0 = \bar{Y}_0$, i.e., the starting values are exact. Thus,

(17) $$AY_0 = \bar{Y}_1 - \left[h^{k+1}\big/(k+1)!\right]\left[\bar{y}^{(k+1)}\left(\zeta_1^0\right), \binom{k+1}{1}\bar{y}^{(k+1)}\left(\zeta_1^1\right), \ldots \binom{k+1}{k}\bar{y}^{(k+1)}\left(\zeta_1^k\right)\right]^T$$

where

$$\zeta_1^j \in (0,h),\ j = 0,1,2,\ldots,\ k$$

(18) $$y(h) = \bar{y}(h) - h^{k+1}\,\bar{y}^{(k+1)}\left(\zeta_1^0\right)\big/(k+1)!$$

$$+ \frac{1}{k+1}\left\{ hf\left(h,\bar{y}(h) - h^{k+1}\,\bar{y}^{(k+1)}\left(\zeta_1^0\right)\ (k+1)!\right)\right.$$

$$\left. - \left[hf(h,\bar{y}(h)) - (k+1)h^{k+1}\,\bar{y}^{(k+1)}\left(\zeta_1^1\right)\big/(k+1)!\right]\right\}$$

$$= \bar{y}(h) + \left[h^{k+1}\big/(k+1)!\right]\left[\bar{y}^{(k+1)}\left(\zeta_1^1\right) - \bar{y}^{k+1}\left(\zeta_1^0\right)\right]$$

$$- \frac{1}{k+1}h^{k+2}\,f_y\,(h,\theta)\bar{y}^{(k+1)}\left(\zeta_1^0\right)\big/(k+1)!$$

$$= \bar{y}(h) + \rho h^{k+2}\bar{y}^{(k+2)}(\psi)\big/(k+1)! - \frac{1}{k+1}h^{k+2}f_y(h,\theta_1)$$

$$\cdot\,\bar{y}^{(k+1)}\left(\zeta_1^0\right)\big/(k+1)!$$

where it follows from the derivation of Equation (15) that

(19) $$\rho\bar{y}^{k+2}(\psi_1) = \bar{y}^{(k+2)}(\psi_1')\big/\left[(k+1)(k+2)\right]$$

$$\psi_1,\ \psi_1' \in (0,h) \text{ and } \theta_1 = \bar{y}(h) - \lambda_1 h^{k+1}\,\bar{y}^{(k+1)}\left(\zeta_1^0\right)\big/(k+1)!,$$

with $0 < \lambda_1 < 1$

(20) $$\therefore\ e^{(0)}(h) = \bar{y}(h) - y(h) = 0(h^{k+2})$$

It should be noted that the third term in the R.H.S. of Equation (18) above can be made as small in absolute value as desired for sufficiently small h by corrector iteration. Similarly, it follows that

(21) $$e^{(1)}(h) = h\bar{y}^{(1)}(h) - hy^{(1)}(h) = 0(h^{k+2}),$$

However, since $\ell_j < \binom{k+1}{j}/(k+1)$ for $j \geq 2$, it follows, by the same reasoning, that

(22)
$$e^{(j)}(h) = \frac{-h^{k+1}}{(k+1)!} \binom{k+1}{j} \bar{y}^{(k+1)}\left(\zeta_1^0\right)$$
$$+ \ell_j \frac{h^{k+1}}{(k+1)!} \left(\binom{k+1}{1} \bar{y}^{(k+1)}\left(\zeta_1^1\right)\right)$$
$$+ O(h^{k+2})$$
$$= O(h^{k+1})$$
$$(j = 2, 3, \ldots, k)$$

Proceeding from h to 2h, it follows that the second-step equation takes the form

(23)
$$y(2h) = \bar{y}(2h)$$
$$+ \left(- h^{k+1}/(k+1)!\right) \sum_{j=2}^{k} \left[1 - j/(k+1)\right] \binom{k+1}{j} \bar{y}^{(k+1)}\left(\zeta_1^j\right)$$
$$+ \left(h^{k+1}/(k+1)!\right) \bar{y}^{(k+1)}\left(\zeta_1^1\right) \ell_j [k+1] \sum_{j=2}^{k} \left[1 - j/(k+1)\right]$$
$$+ O(h^{k+2}) = \bar{y}(2h) + O(h^{k+1})$$

Hence, as was to be shown,

(24)
$$e^{(0)}(2h) = O(h^{k+1})$$

and it is easily seen that the order of $e^{(0)}(2h)$ cannot be raised to $O(h^{k+2})$ by corrector iteration. This result immediately generalizes to any multivalue method characterized by nonzero stabilizing coefficients. That is, every such initial-step maximal-polynomial-degree method must suffer a decrease of one in the order of the error in proceeding from h to 2h. This phenomenon will, of course, not be observable in the behavior of non-maximal-degree initial-step methods, where the initial single-step error is already $O(h^{k+1})$.

Next is exhibited a method for preserving order in the fixed step-size context by appropriately changing the degree maximizing coefficient from step-to-step for not more than k+1 steps. This order-preservation method will be a consequence of the following "error mop-up lemma" and a fixed-point theorem.

Lemma(*)

For fixed h and N let $\alpha_0, \alpha_1, \ldots, \alpha_N$ be N real numbers, not all zero, such that $\sum_{i=0}^{N} \alpha_i = 0$. Let $y(x) \in C^{k+2}[0, Nh]$ with $x_i \in [0, Nh]$, $i = 0, 1, 2 \ldots, N$.

Then

$$\sum_{i=0}^{N} \alpha_i y^{(k+1)}(x_i) = 0(h) \tag{25}$$

Proof

For each $i = 0, 1, 2, \ldots N$

$$y^{(k+1)}(x_i) = y^{(k+1)}(0) + x_i y^{(k+2)}(\xi_i) \tag{26}$$

$$\therefore \sum_{i=0}^{N} \alpha_i y^{(k+1)}(x_i) = y^{(k+1)}(0) \sum_{i=0}^{N} \alpha_i + \sum_{i=0}^{N} \alpha_i x_i y^{(k+2)}(\xi_i) \tag{27}$$

$$= \sum_{i=0}^{N} \alpha_i x_i y^{(k+2)}(\xi_i) = 0(h)$$

(*) This simple proof kindly provided by Dr. C. W. Gear

The following observation is now made by way of Equations (17) through (23). Suppose S_n is defined as the vector of $O(h^{k+1})$ constant error coefficient sums at step n. Thus, for example, from Equation (22)

$$(28)\qquad S_1 = \left[h^{k+1}/(k+1)!\right]\begin{bmatrix} 0 \\ 0 \\ (k+1)\,\ell_2 - \binom{k+1}{2} \\ \vdots \\ (k+1)\,\ell_k - \binom{k+1}{k} \end{bmatrix},$$

and from Equation (23)

$$(29)\qquad S_2 = \left[h^{k+1}/(k+1)!\right]\begin{bmatrix} \sum_{j=2}^{k} 1 - \left[j/(k+1)\right]\left[(k+1)\,\ell_j - \binom{k+1}{j}\right] \\ \vdots \end{bmatrix}$$

Then, it follows by induction on n that, up to the proportionality constant, $h^{k+1}/(k+1)!$,

$$(30)\qquad S_{n+1} = \left[I - \ell e_1^T\right]\left[AS_n - \left\{\binom{k+1}{i}\right\}\right],$$

where

$$\left\{\binom{k+1}{i}\right\} = \begin{bmatrix} 1 \\ \binom{k+1}{1} \\ \binom{k+1}{2} \\ \vdots \\ \binom{k+1}{k} \end{bmatrix}$$

Thus, if at step n+1 it is possible to choose an $\ell_0^{(n+1)}$ so that

$$e_0^T AS_n = e_0^T AS_n - 1 + \left(k+1-e_1^T AS_n\right) \ell_0^{(n+1)} = 0,$$

the $e^{(0)}((n+1)h)$ $0(h^{k+1})$ constant error coefficients sum to zero, the lemma applies and $e^{(0)}((n+1)h) = 0(h^{k+2})$. The following lemma shows that if such a choice of the $\ell_0^{(n+1)}$ is possible, then the sequence terminates after not more than k+1 steps; i.e., $\ell_0^{(k)} = \ell_0^{(k+1)} = \ldots.$

Lemma

If $k+1 - e_1^T AS_n \neq 0$ for $n = 0, 1, 2, \ldots, k$, then S_k is the unique fixed point of the mapping:

(31) $$\varphi(S) = \left(\Pi - \ell^* e_1^T\right)\left(AS - \left\{\binom{k+1}{i}\right\}\right),$$

where

$$\Pi = \begin{bmatrix} 0 & \ldots 0 \ldots & 0 \\ \cdot & & \\ \cdot & & \\ 0 & I & \\ \cdot & & \\ \cdot & & \\ 0 & & \end{bmatrix},$$

and

$$\ell^* = \begin{bmatrix} 0 \\ 1 \\ \ell_2 \\ \cdot \\ \cdot \\ \ell_k \end{bmatrix}.$$

This system of linear equations can be expanded, component by component, to yield the system of k-1 equations

(32) $$\sum_{i=1}^{k-j} \binom{j+i}{j} s_{j+i} - \ell_j \sum_{r=2}^{k} r s_r = \binom{k+1}{j} - (k+1)\ell_j$$

$$(j = 2, 3, \ldots, k-1)$$

and

$$- \ell_k \sum_{r=2}^{k} r s_r = (k+1)(1-\ell_k)$$

Which, since $\ell_k \neq 0$, has the unique solution

(33)
$$s_0 = 0$$

$$s_1 = 0$$

$$\begin{bmatrix} s_2 \\ s_3 \\ \cdot \\ \cdot \\ \cdot \\ s_k \end{bmatrix} = \binom{j+i}{j}^{-1} \left[\binom{k+1}{j} - (k+1)\ell_j/\ell_k \right]$$

Proof

Suppose $k+1 - e_1^T A S_n \neq 0$ for $n = 0, 1, 2, \ldots, k$. Then, for each n, S_n has zero first and second components and .

(34)
$$S_{k+1} - S_k = \Pi(S_{k+1} - S_k)$$

$$= \Pi(I - \ell^{k+1} e_1^T)A\Pi(S_k - S_{k-1}) - \Pi(\ell^{k+1} - \ell^k)e_1^T\binom{k+1}{i}$$

$$= (\Pi - \ell^* e_1^T)A\Pi(S_k - S_{k-1}) = (\Pi - \ell^* e_1^T)A(S_k - S_{k-1})$$

Hence, $S_{k+1} - S_k = \left[\Pi(I - \ell^* e_1^T)A\right]^{k+1} S_1 = 0$ since the characteristic polynomial of $\pi(I - \ell^* e_1^T)A$ is x^{k+1}.

Hence,

(35)
$$S_k = S_{k+1} = \ldots. = S.$$

Thus, if $k+1 - e_1^T A S_n \neq 0$ for all $n \leq k$

(36)
$$\ell_0^{(n+1)} = (1 - e_0^T A S_n)/(k+1 - e_1^T A S_n)$$

provides the required ℓ_0 sequence.

That the above lemma is not vacuous in content can be shown for the cases $k = 2, 3, 4, 5$, by direct computation, yielding the following (k, 1) table:

$k/\ell_0^{(n)}$	$\ell_0^{(1)}$	$\ell_0^{(2)}$	$\ell_0^{(3)}$	$\ell_0^{(4)}$	$\ell_0^{(5)}$
2	1/3	5/12			
3	1/4	11/30	3/8		
4	1/5	469/1440	959/2760	251/720	
5	1/6	5659/19440	10871/33480	28249/85680	95/288

It should be noted that the terminating coefficients 5/12, 3/8, etc. are identical to Gear's. Thus, naturally, the following conjecture: For each k and for each p a terminating order-optimizing coefficient sequence can be constructed. The terminal coefficient is equal to Gear's which is, in turn, identical to the classical Adams-Moulton for p=1.

The above conclusions and the conjecture apply in the fixed-step context. In the general case, if the ratio of new step size to old is ρ_n in going from step n to n+1, then it is easy to show that the order-preserving coefficient formula generalizes to:

$$(37) \qquad \ell_0^{(n+1)} = \left[1 - e_0^T A D_n S_n / \rho_n^{k+1}\right] \Big/ \left[k+1 - e_1^T A D_n S_n / \rho_n^{k+1}\right]$$

where

$$D_n = \begin{bmatrix} 1 & & & 0 \\ & \rho_n & & \\ 0 & & \rho_n^2 & \\ & & & \ddots \\ & & & & \rho_n^k \end{bmatrix},$$

provided the denominator does not vanish. Since the denominator vanishes for at most k+1 distinct ρ_n it is thus always possible to find a suitable ρ_n for step-size expansion or contraction which preserve both stability and maximum order.

CONCLUSION

It follows, from the results, that the notion of equivalence among numerical methods in the sense of Gear and Descloux is not applicable when starting values are sufficiently accurate or in the presence of stepsize change. However, the important, fundamental core of Gear's work, the matrix theoretical construction of maximally stable k+1 value methods, remains intact. This foundation has been built upon by removing some logical inconsistencies. The removal of these inconsistencies has revealed the intrinsic properties of a family of truly optimally stable, one-step,

maximum p.d. methods believed to have great unexplored potential for accuracy, stability, speed, error control, ease of interpolation, and implementability. Results on the development of starting, error estimation, and control procedures and experimental results with variable-step versions will be described in a later paper.

ACKNOWLEDGEMENTS

I would like to express my appreciation to Dr. J. Dyer, and my colleagues, Drs. I. Tarnove, D. Pope and D. Carta, for their invaluable advice and encouragement in the preparation of this paper. Also, a special thanks to Ralph Kahn for his programming support.

REFERENCES

1. Descloux, J., "Note on a Paper by Nordsieck, "Department of Computer Science Report No. 131, University of Illinois, Urbana, Illinois.

2. Gear, C. W., "The Numerical Integration of Ordinary Differential Equations", Math. Comp., 21, pp 146-156.

3. Nordsieck, A. (1962), "On the Numerical Integration of Ordinary Differential Equations," Math. Comp., 16, pp 22-49.